Basic Concepts of Thermomechanics

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0. Introduction

This paper is intended to serve as a model for the first few chapters of future textbooks on continuum mechanics and continuum thermomechanics. It may be considered an update of the paper *Lectures on the Foundations of Continuum Mechanics and Thermodynamics* [N2] by one of us (W.N.), published in 1973, and an elaboration of topics treated in Part 3, entitled *Updating the Non-Linear Field Theories of Mechanics*, of the booklet [FC] by W.N.. [1]

The present paper differs from most existing textbooks on the subject in several important respects:

1) It uses the mathematical infrastructure based on sets, mappings, and families, rather than the infrastructure based on variables, constants, and parameters. (For a detailed explanation, see *The Conceptual Infrastructure of Mathematics* by W.N. [N1].)

2) It is completely coordinate-free and $\mathbb{R}^n$-free when dealing with basic concepts.

3) It does not use a fixed *physical space*. Rather, it employs an infinite variety of *frames of reference*, each of which is a Euclidean space. The motivation for avoiding physical space can be found in Part 1, entitled *On the Illusion of Physical Space*, of the booklet [FC]. Here, the basic laws are formulated without the use of a physical space or any external frame of reference.

4) It considers inertia as only one of many external forces and does not confine itself to using only inertial frames of reference. Hence kinetic energy, which is a potential for inertial forces, does not appear separately in the energy balance equation. In particle mechanics, inertia plays a fundamental role and the subject would collapse if it is neglected. Not so in continuum mechanics, where it is often appropriate to neglect inertia, for example when analyzing the motion of toothpaste when it is extruded slowly from a tube.

This paper does not deal with several important issues. For example:

1) It assumes that internal interactions at a distance, both forces and heat transfers, are absent. They should be included in a more inclusive analysis because they are important, for example, in applications of continuum thermomechanics to astrophysics.

2) It takes the basic properties of concepts such as force, stress, energy, heat transfer, temperature, and entropy for granted and it does not deal with the large and important literature that tries to derive them from more primitive assumptions.

3) It does not deal with the description of phase transitions.

4) It does not deal with the description of diffusion, i.e., the intermingling of different substances.

5) It does not deal with the connection between chemical reactions and continuum thermomechanics.

6) It does not deal with electromagnetic phenomena. These are inherently relativistic and hence are not based on the existence of an absolute time. We know of no satisfactory reconciliation of relativity and continuum physics, which needs absolute time. (However, absolute space is present neither in relativity nor in continuum physics as presented here.) What is needed is some non-relativistic approximation for electromagnetism.

We hope that, in the future, the issues just described will be treated in the same spirit as the present paper, in particular by using the mathematical infrastructure based on sets, mappings, and families and without using a fixed physical space,

1. Physical Systems

The concept of a materially ordered set was first introduced by W.N. in the context of an axiomatic foundation of physical systems (see [N2])[2]. The present description is taken from [NS].

[1] However, the present paper introduces important new ideas due to Brian Seguin.

[2] In the mid 1950's W.N. regularly taught courses for engineering students with the titles *Statics* and *Dynamics*. In *Statics*, the students were asked to consider some system (a building, a bridge, or a machine), draw *free-body diagrams*, and apply to each of these the balance of forces and torques. This often gave enough linear equations to determine the stresses in each of the pieces of the system. In *Dynamics*, the students were asked to apply the same procedure as in *Statics* except that inertial forces are taken into account. This often led to linear differential equations. W.N. then wondered what the underlying conceptual structure of all this was. The material in this section is what he came up with.

Here Ω is considered to consist of the whole system and all of its parts. Given $a, b \in \Omega$, $a \prec b$ is read "a is a part of b". The maximum ma is the "material all", i.e. the whole system, and the minimum mn is the "material nothing". The $\inf\{a, b\}$ is the overlap of a and b, and a^{rem} is the part of the whole system ma that remains after a has been removed. With this in mind, the two conditions (MO3) and (MO4) below are very natural.

Definition 1. *An ordered set Ω with order $\prec$ is said to be* **materially ordered** *if the following axioms are satisfied:*

(MO1) *Ω has a maximum* ma *and a minimum* mn.

(MO2) *Every doubleton has an infimum.*

(MO3) *For every $p \in \Omega$ there is exactly one member of Ω, denoted by p^{rem}, such that* $\inf\{p, p^{\text{rem}}\} = \text{mn}$ *and* $\sup\{p, p^{\text{rem}}\} = \text{ma}$.

(MO4) $(\inf\{p, q^{\text{rem}}\} = \text{mn}) \implies p \prec q \quad \text{for all} \quad p, q \in M$.

The mapping $\text{rem} := (p \mapsto p^{\text{rem}}) : \Omega \longrightarrow \Omega$ *is called the* **remainder mapping** *in Ω.*

Theorem 1: *Let Ω be a materially ordered set. Then Ω has the structure of a Boolean algebra with*

$$p \wedge q := \inf\{p, q\} \quad \text{and} \quad p \vee q := \sup\{p, q\} \qquad \text{for all} \quad p, q \in \Omega, \tag{1.1}$$

which means that the following relations are valid:

$$\text{mn} = \text{ma}^{\text{rem}} \ , \tag{1.2}$$

$$p \wedge \text{ma} = p, \tag{1.3}$$

$$p \wedge p^{\text{rem}} = \text{mn} \ , \tag{1.4}$$

$$(p^{\text{rem}})^{\text{rem}} = p \ , \tag{1.5}$$

$$p \wedge q = q \wedge p \ , \tag{1.6}$$

$$(p \wedge q) \wedge r = p \wedge (q \wedge r) \ , \tag{1.7}$$

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \ , \tag{1.8}$$

$$(p \vee q)^{\text{rem}} = p^{\text{rem}} \wedge q^{\text{rem}} \ , \tag{1.9}$$

valid for all $p, q, r \in M$.

The symbol $p \wedge q$ is read as p **meet** q, and the symbol $p \vee q$ is read as p **join** q.

All of the formulas above remain valid if every join and meet as well as ma and mn are interchanged. We will refer to the new version of an equation obtained in this way as the *dual* of the original equation. For example, the dual of (1.4) is $p \vee p^{\text{rem}} = \text{ma}$.

All of the formulas (1.2) - (1.7), and also their duals, are intuitively very plausible. The formulas (1.8) and (1.9) and their duals are less plausible. The proofs are highly non-trivial. The best version of these is given in [NS].

Theorem 2: *Let Ω be a materially ordered set and $p \in \Omega$ be given. Then $\Omega_p := \{q \in \Omega \mid q \prec p\}$ is a materially ordered set and the remainder mapping in Ω_p is given by*

$$\text{rem}_p := (a \mapsto a^{\text{rem}} \wedge p) \ . \tag{1.10}$$

The proof is easy.

2. Additive Mappings and Interactions

Let Ω be a materially ordered set and $\mathcal{W}$ a linear space. We say that the parts p and q are **separate** if $p \wedge q = \text{mn}$. We use the notation

$$(\Omega^2)_{\text{sep}} := \{(p, q) \in \Omega^2 \mid p \wedge q = \text{mn}\} \ . \tag{2.1}$$

Definition 2. *A function* $\mathbf{H} : \Omega \longrightarrow \mathcal{W}$ *is said to be* **additive** *if*

$$\mathbf{H}(p \vee q) = \mathbf{H}(p) + \mathbf{H}(q) \quad \text{for all} \quad (p, q) \in (\Omega^2)_{\text{sep}} \ . \tag{2.2}$$

A function $\mathbf{I} : (\Omega^2)_{\text{sep}} \longrightarrow \mathcal{W}$ *is said to be an* **interaction** *in* Ω *if, for all* $p \in \Omega$*, both*

$$\mathbf{I}(\ \cdot\ , p^{\text{rem}}) : \Omega_p \longrightarrow \mathcal{W} \ \text{ and } \ \mathbf{I}(p^{\text{rem}},\ \cdot\ ) : \Omega_p \longrightarrow \mathcal{W}$$

are additive.

The **resultant** $\text{Res}_{\mathbf{I}} : \Omega \to \mathcal{W}$ *of a given interaction* $\mathbf{I}$ *in* Ω *is defined by*

$$\text{Res}_{\mathbf{I}}(p) := \mathbf{I}(p, p^{\text{rem}}) \quad \text{for all} \ \ p \in \Omega \quad . \tag{2.3}$$

We say that a given interaction is **skew** *if*

$$\mathbf{I}(q, p) = -\mathbf{I}(p, q) \ \text{ for all } (p, q) \in (\Omega^2)_{\text{sep}} \ . \tag{2.4}$$

Remark 1: The concept of an interaction is an abstraction. Its values may have the interpretation of forces, torques, or heat transfers. In most of the past literature these cases were treated separately even though much of the underlying mathematics is the same for all. Thus, this abstraction, like most others, is a labor saving device. ▮

Theorem 3: *An interaction is skew if and only if its resultant is additive.*

Proof: Let $(p, q) \in (\Omega^2)_{\text{sep}}$ be given, so that $p \wedge q = \text{mn}$. Using some of the rules (1.2)-(1.9), a simple calculation shows that

$$p^{\text{rem}} = q \vee (p \vee q)^{\text{rem}} \ \text{ and } \ \text{mn} = q \wedge (p \vee q)^{\text{rem}} \ , \tag{2.5}$$

so that $(q, (p \vee q)^{\text{rem}}) \in (\Omega^2)_{\text{sep}}$.

Using the additivity of $\mathbf{I}(p, \cdot) : \Omega_{p^{\text{rem}}} \longrightarrow \mathcal{W}$ it follows that

$$\text{Res}_{\mathbf{I}}(p) = \mathbf{I}(p, p^{\text{rem}}) = \mathbf{I}(p, q) + \mathbf{I}(p, (p \vee q)^{\text{rem}}). \tag{2.6}$$

Interchanging the roles of p and q we find that

$$\text{Res}_{\mathbf{I}}(q) = \mathbf{I}(q, q^{\text{rem}}) = \mathbf{I}(q, p) + \mathbf{I}(q, (q \vee p)^{\text{rem}}). \tag{2.7}$$

Adding (2.6) and (2.7), using the additivity of $\mathbf{I}(\cdot, (p \vee q)^{\text{rem}})$, and then (2.3) with p replaced by $p \vee q$, we obtain

$$\text{Res}_{\mathbf{I}}(p) + \text{Res}_{\mathbf{I}}(q) - \text{Res}_{\mathbf{I}}(p \vee q) = \mathbf{I}(p, q) + \mathbf{I}(q, p) \ , \tag{2.8}$$

from which the assertion follows. ▮

3. Continuous Bodies

In order to define a continuous body system two classes must be specified. One being the class Fr of all subsets of three-dimentional Euclidean spaces that are possible regions that a body system can occupy. Intuitively, the term “body” suggests that the regions it can occupy are connected. We do not assume this but we will use the term “body” rather than “body system” from now on. The other being the class Tp of mappings which are possible changes of placement of a body.

It is useful to take Fr to be the class of **fit regions** introduced in [NV]. Roughly speaking, a fit region is an open bounded subset of a Euclidean space whose boundary fails to have an exterior normal only at exceptional points. Let a Euclidean space $\mathcal{E}$, with translation space $\mathcal{V}$, be given. We denote by $\text{Fr}\,\mathcal{E}$ the set of all fit regions in $\mathcal{E}$. Let $\mathcal{A} \in \text{Fr}\,\mathcal{E}$ be given. We denote the set of points in which there is an exterior normal to $\mathcal{A}$ by $\text{Rby}\mathcal{A}$ and call it the **reduced boundary** of $\mathcal{A}$. Let

$$\mathbf{n}_{\mathcal{A}} : \text{Rby}\mathcal{A} \longrightarrow \text{Usph}\mathcal{V} \tag{3.1}$$

be the mapping that assigns to each point of the reduced boundary the exterior unit normal. Let $\mathbf{C} : \mathcal{A} \longrightarrow \text{Lin}\,(\mathcal{V}, \mathcal{W})$ be a mapping that assigns to each point $x \in \mathcal{A}$ a linear mapping from $\mathcal{V}$ to some linear space $\mathcal{W}$. Then the divergence theorem holds, namely

$$\int_{\mathrm{Rdy}\mathcal{A}} \mathbf{C}\mathbf{n}_{\mathcal{A}} = \int_{\mathcal{A}} \mathrm{div}\mathbf{C} \ . \tag{3.2}$$

The class Tp consists of all mappings λ with the following properties:

($\mathbf{T}_1$) λ is an invertible mapping whose domain Domλ and range Rng λ are fit regions in Euclidean spaces Dspλ and Rspλ, which are called the **domain-space** and **range-space** of λ, respectively.

($\mathbf{T}_2$) There is a C^2-diffeomorphism $\phi : \mathrm{Dsp}\lambda \longrightarrow \mathrm{Rsp}\lambda$ such that $\lambda = \phi|_{\mathrm{Dom}\,\lambda}^{\mathrm{Rng}\,\lambda}$.

The class Tp, whose elements are called **transplacements**, is stable under composition in the sense that for any $\lambda, \gamma \in$ Tp with Dom $\lambda =$ Rng γ we have $\lambda \circ \gamma \in$Tp. It is also stable under inversion in the sense that if $\lambda \in$Tp then $\lambda^{\leftarrow} \in$Tp.

Assume that a set $\mathcal{B}$ is given. We say that a function $\delta : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{P}$ is a **fit Euclidean metric** on $\mathcal{B}$ if it makes $\mathcal{B}$ isometric to some fit region in some Euclidean space. As shown in Section 6 of Part 2 of [FC], it is then possible to use δ to imbed[3] $\mathcal{B}$ into a Euclidean space $\mathcal{E}_\delta$ constructed from $\mathcal{B}$ using δ. The imbedding imb_δ is invertible with Dom $\mathrm{imb}_\delta = \mathcal{B}$ and $\mathcal{B}_\delta := \mathrm{Rng}\,\mathrm{imb}_\delta \in \mathrm{Fr}\,\mathcal{E}_\delta$ such that

$$\delta(X, Y) = \mathrm{dist}(\mathrm{imb}_\delta(X), \mathrm{imb}_\delta(Y)) \quad \text{for all} \quad X, Y \in \mathcal{B} \ , \tag{3.3}$$

where dist denotes the Euclidean distance in $\mathcal{E}_\delta$. We call $\mathcal{E}_\delta$ the **imbedding space** for δ and imb_δ the **imbedding mapping** for δ.

Definition 3. *A* **continuous body** $\mathcal{B}$ *is a set endowed with structure by the specification of a non-empty set* Conf $\mathcal{B}$*, whose elements are called* **configurations** *of* $\mathcal{B}$*, satisfying the following requirements:*

($\mathbf{B}_1$) *Every* $\delta \in \mathrm{Conf}\,\mathcal{B}$ *is a fit Euclidean metric.*

($\mathbf{B}_2$) *For all* $\delta, \epsilon \in \mathrm{Conf}\,\mathcal{B}$ *the mapping* $\lambda := \mathrm{imb}_\delta \circ \mathrm{imb}_\epsilon^{\leftarrow}$ *is a transplacement, with* $\mathrm{Dsp}\lambda = \mathcal{E}_\epsilon$ *and* $\mathrm{Rsp}\lambda = \mathcal{E}_\delta$.

($\mathbf{B}_3$) *For every* $\delta \in \mathrm{Conf}\,\mathcal{B}$ *and every transplacement* λ *such that* $\mathrm{Dom}\,\lambda = \mathrm{Rng}\,\mathrm{imb}_\delta$*, the function* $\epsilon : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{P}$*, defined by*

$$\epsilon(X, Y) := \mathrm{dist}(\lambda(\mathrm{imb}_\delta(X)), \lambda(\mathrm{imb}_\delta(Y)) \quad \text{for all} \quad X, Y \in \mathcal{B} \ , \tag{3.4}$$

is a fit Euclidean metric that belongs to Conf $\mathcal{B}$.

The elements of $\mathcal{B}$ are called **material points**.

For the rest of this paper, we assume that a non-empty continuous body $\mathcal{B}$ is given. The imbeddings of $\mathcal{B}$ endow $\mathcal{B}$ with the structure of a three-dimensional C^2-manifold. Thus $\mathcal{B}$ is a topological space and at each material point $X \in \mathcal{B}$ there is a tangent space $\mathcal{T}_X$ which is a three-dimensional linear space. The space $\mathcal{T}_X$ is called the (infinitesimal) **body element** of $\mathcal{B}$ at X since it is the precise mathematical representation of what many engineers refer to as an "infinitesimal element" of the body. Note that the tangent spaces are not inner-product spaces and hence the dual $\mathcal{T}_X^*$ of $\mathcal{T}_X$ will come up frequently.

Let $\delta \in \mathrm{Conf}\,\mathcal{B}$ be given. Let $\mathcal{E}_\delta$ denote the corresponding imbedding space, with translation space $\mathcal{V}_\delta$, and let imb_δ denote the imbedding mapping for δ. The gradient of imb_δ at a material point $X \in \mathcal{B}$,

$$\mathbf{I}_\delta(X) := \nabla_X \mathrm{imb}_\delta \in \mathrm{Lis}\,(\mathcal{T}_X, \mathcal{V}_\delta) \ , \tag{3.5}$$

is a linear isomorphism from the body element $\mathcal{T}_X$ to the translation space $\mathcal{V}_\delta$ of the imbedding space. It can be used to define

$$\mathbf{G}_\delta(X) := \mathbf{I}_\delta^\top(X)\mathbf{I}_\delta(X) \in \mathrm{Pos}^+(\mathcal{T}_X, \mathcal{T}_X^*) \ , \tag{3.6}$$

which we call the **configuration of the body element** $\mathcal{T}_X$ since it is the localization of the global configuration $\delta \in \mathrm{Conf}\,\mathcal{B}$.

Remark 2: In the literature on differential geometry the mappings $\mathbf{I}_\delta := (X \mapsto \mathbf{I}_\delta(X))$ and $\mathbf{G}_\delta := (X \mapsto \mathbf{I}_\delta(X))$ are cross-sections of appropriately defined fiber bundles. A configuration δ gives $\mathcal{B}$ the structure

[3]Only the existence of such an imbedding is important here, not the details of its construction.

of a Riemannian manifold. The cross-section $\mathbf{G}_\delta$ is often called the *metric tensor field* for the Euclidean-Riemannian structure defined by δ. ▮

Remark 3: The class Tp specified above corresponds to materials without constraints. If one wished to describe materials with constraints then the class Tp would have to be restricted. For example, if one also requires that the transplacements are volume preserving then Tp makes the body incompressible. ▮

Consider the set $\Omega_\mathcal{B}$ defined by

$$\Omega_\mathcal{B} := \{\mathcal{P} \in \mathrm{Sub}\mathcal{B} \mid \mathrm{imb}_{\delta>}(\mathcal{P}) \in \mathrm{Fr} \text{ for some } \delta \in \mathrm{Conf}\,\mathcal{B}\} \ . \tag{3.7}$$

It follows from (T_1) and (B_2) that if $\mathrm{imb}_{\delta>}(\mathcal{P}) \in \mathrm{Fr}$ for some $\delta \in \mathrm{Conf}\mathcal{B}$ then $\mathrm{imb}_{\delta>}(\mathcal{P})$ is a fit region for every configuration δ. If $\mathcal{P}$ is an element of $\Omega_\mathcal{B}$ then the set

$$\mathrm{Conf}\mathcal{P} := \{\delta|_{\mathcal{P}\times\mathcal{P}} \mid \delta \in \mathrm{Conf}\,\mathcal{B}\} \tag{3.8}$$

endows $\mathcal{P}$ with the structure of a continuous body. For this reason we sometimes call such $\mathcal{P}$ **parts** or **sub-bodies** of $\mathcal{B}$.

The set $\Omega_\mathcal{B}$ is *materially ordered*, by inclusion in the sense of Definition 1 and hence, by Theorem 1, it has the structure of a Boolean algebra. We have

$$\mathcal{P} \wedge \mathcal{Q} := \mathcal{P} \cap \mathcal{Q} \ , \tag{3.9}$$

$$\mathcal{P} \vee \mathcal{Q} := \mathrm{Int}\ \mathrm{Clo}(\mathcal{P} \cup \mathcal{Q}) \ , \tag{3.10}$$

$$\mathcal{P}^{\mathrm{rem}} := \mathrm{Int}(\mathcal{B}\backslash\mathcal{P}) \ . \tag{3.11}$$

The proof of this is highly non-trivial result can be found in [NV]. One should think of $\mathcal{P} \wedge \mathcal{Q}$ as the common part of $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{P} \vee \mathcal{Q}$ as the part obtained by merging $\mathcal{P}$ and $\mathcal{Q}$, and $\mathcal{P}^{\mathrm{rem}}$ as the part of $\mathcal{B}$ left when $\mathcal{P}$ is taken away.

4. Frames of Reference and Placements

When dealing with the behavior of a continuous body in an environment it is useful to employ a frame of reference. Such frames are represented mathematically by three-dimentional Euclidean spaces. We call Euclidean spaces that represent frames of reference **frame-spaces**.

Definition 4. *Let μ be an invertible mapping with $\mathrm{Dom}\,\mu = \mathcal{B}$ and $\mathcal{B}_\mu := \mathrm{Rng}\,\mu \in \mathrm{Fr}$. We say that μ is a* **placement of** $\mathcal{B}$ *if $\mathrm{imb}_\delta \circ \mu^{\leftarrow}$ is a transplacement for every $\delta \in \mathrm{Conf}\,\mathcal{B}$. The Euclidean space in which $\mathrm{Rng}\,\mu$ is a fit region is called the* **range-space** *of μ and will be denoted by $\mathrm{Frm}\,\mu$. We denote the translation space of $\mathrm{Frm}\,\mu$ by $\mathrm{Vfr}\,\mu$. We denote the set of all placements of $\mathcal{B}$ by $\mathrm{Pl}\,\mathcal{B}$.*

The following facts are easy consequences of Definitions 3 and 4:

($\mathbf{P}_1$) For all $\kappa, \gamma \in \mathrm{Pl}\,\mathcal{B}$ we have $\kappa \circ \gamma^{\leftarrow} \in \mathrm{Tp}$.

($\mathbf{P}_2$) For every $\kappa \in \mathrm{Pl}\,\mathcal{B}$ and $\lambda \in \mathrm{Tp}$ such that $\mathrm{Rng}\,\kappa = \mathrm{Dom}\,\lambda$ we have $\lambda \circ \kappa \in \mathrm{Pl}\,\mathcal{B}$.

Of course, every imbedding imb_δ, $\delta \in \mathrm{Conf}\mathcal{B}$, is a placement, but not every placement is an imbedding. It follows from (P_2) that, in Definition 4, "every" can be replaced by "some" without changing the validity.

Let a placement $\mu : \mathcal{B} \longrightarrow \mathcal{B}_\mu \subset \mathrm{Frm}\mu$ be given. We then define $\delta_\mu : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{P}^\times$ by

$$\delta_\mu(X, Y) := \mathrm{dist}(\mu(X), \mu(Y)) \quad \text{for all} \quad X, Y \in \mathcal{B} \ , \tag{4.1}$$

where dist is the Euclidean distance in $\mathrm{Frm}\,\mu$. It follows from (B_3) and Definition 4 that δ_μ is Euclidean metric and, in fact, a configuration of $\mathcal{B}$. We call δ_μ the **configuration induced by the placement** μ. [4]

Now let a configuration $\delta \in \mathrm{Conf}\mathcal{B}$ be given. Let μ and μ' be two placements that induce the same configuration δ. Since $\delta = \delta_\mu = \delta_{\mu'}$, it follows from (4.1) that $\mu' \circ \mu^{\leftarrow}$ is a Euclidean isometry. Since there

[4] In 1958, W.N. introduced the unfortunate terms "configuration" and "deformation" for what are called "placement" and "transplacement" here. He apologizes. Since these old terms were used in [NLFT], they were widely accepted, and we are now in the ironic position to fight against a terminology that W.N. introduced.

are infinitely many Euclidean isometries with domain $\mathcal{B}_\mu$ there are infinitely many placements that induce the given configuration. In particular, in the case when $\mu' := \mathrm{imb}_\delta$,

$$\alpha := \mathrm{imb}_\delta \circ \mu^{\leftarrow} : \mathcal{B}_\mu \longrightarrow \mathcal{B}_\delta \tag{4.2}$$

is a Euclidean isometry. Its (constant) gradient

$$\mathbf{Q} := \nabla(\mathrm{imb}_\delta \circ \mu^{\leftarrow}) \in \mathrm{Orth}(\mathrm{Vfr}\,\mu, \mathcal{V}_\delta) \tag{4.3}$$

is an inner-product isomorphism.

Define

$$\mathbf{M}_\mu(X) := \nabla_X \mu \in \mathrm{Lis}\,(\mathcal{T}_X, \mathrm{Vfr}\,\mu) \qquad \text{for all} \quad X \in \mathcal{B}\ . \tag{4.4}$$

In view of (3.5), it follows from (4.3) and the chain rule that

$$\mathbf{I}_\delta = \mathbf{Q}\mathbf{M}_\mu, \tag{4.5}$$

and hence, by (3.6), that the configuration of the body element $\mathcal{T}_X$ induced by δ is given by

$$\mathbf{G}_\delta(X) = \mathbf{M}_\mu(X)^\top \mathbf{M}_\mu(X) \in \mathrm{Pos}^+(\mathcal{T}_X, \mathcal{T}_X^*) \quad \text{for all} \;\; X \in \mathcal{B}\ . \tag{4.6}$$

For a given placement μ we use the notation

$$\mathcal{P}_\mu := \mu_>(\mathcal{P}) \quad \text{for all} \;\; \mathcal{P} \in \Omega_\mathcal{B}\ . \tag{4.7}$$

Given $(\mathcal{P}, \mathcal{Q}) \in (\Omega_\mathcal{B})^2_{\mathrm{sep}}$ we define the **reduced contact** of $(\mathcal{P}, \mathcal{Q})$ in μ by

$$\mathrm{Rct}_\mu(\mathcal{P}, \mathcal{Q}) := \mathrm{Rby}\mathcal{P}_\mu \cap \mathrm{Rby}\mathcal{Q}_\mu\ . \tag{4.8}$$

Let another placement μ' be given. Let $\mathbf{A} \in \mathrm{Orth}(\mathrm{Vfr}\,\mu', \mathrm{Vfr}\,\mu)$ be an inner-product preserving isomorphism from $\mathrm{Vfr}\,\mu'$ to $\mathrm{Vfr}\,\mu$. Put

$$\lambda := \mu \circ \mu'^{\leftarrow}\ . \tag{4.9}$$

Define the **local volume change function** $\rho_{\mu',\mu} : \mathcal{B}_{\mu'} \longrightarrow \mathbb{P}^\times$ by

$$\rho_{\mu',\mu}(x) := |\det(\nabla_x \lambda \mathbf{A})| \quad \text{for all} \;\; x \in \mathcal{B}_{\mu'}\ . \tag{4.10}$$

This definition is independent of which inner-product isomorphism from $\mathrm{Vfr}\,\mu'$ to $\mathrm{Vfr}\,\mu'$ is used. To see this let $\mathbf{A}' \in \mathrm{Orth}(\mathrm{Vfr}\,\mu', \mathrm{Vfr}\,\mu)$ be another such isomorphism from $\mathrm{Vfr}\,\mu'$ to $\mathrm{Vfr}\,\mu$. It is clear that $\mathbf{Q} := \mathbf{A}^{-1}\mathbf{A}' \in \mathrm{Orth}(\mathrm{Vfr}\,\mu')$. Hence, since the determinant of an orthogonal lineon is ± 1, we have

$$|\det(\nabla_x \lambda \mathbf{A}')| = |\det(\nabla_x \lambda \mathbf{A}\mathbf{Q})| = |\det(\nabla_x \lambda \mathbf{A})||\det \mathbf{Q}| = |\det(\nabla_x \lambda \mathbf{A})|\ . \tag{4.11}$$

Since the transplacement λ is of class C^2, $\rho_{\mu',\mu}$ is of class C^1.

5. Time-Families

In much of the rest of this paper we assume that a genuine real interval I, called the *time-interval*, is given. Any family indexed on I will be called a **time-family**. In some cases, each of the terms f_t of the family belong to a given set $\mathcal{S}$. In this case, the family can be identified with a mapping $f : I \longrightarrow \mathcal{S}$, so that

$$f(t) := f_t \quad \text{for all} \;\; t \in I\ . \tag{5.1}$$

If $\mathcal{S}$ is a Euclidean space or linear space, it makes sense to consider the case when f is of class C^1 or C^2 and then define the time-families $(f_t^\bullet \mid t \in I)$ or $(f_t^{\bullet\bullet} \mid t \in I)$ by

$$f_t^\bullet := f^\bullet(t) \;\; \text{or} \;\; f_t^{\bullet\bullet} := f^{\bullet\bullet}(t) \quad \text{for all} \;\; t \in I\ . \tag{5.2}$$

In some cases, one deals with a time-family $(\mathcal{A}_t \mid t \in I)$ of sets and considers a time-family $(g_t \mid t \in I)$ of mappings $g_t : \mathcal{A}_t \longrightarrow \mathcal{S}$ with values in a set $\mathcal{S}$. Putting $\mathcal{M} := \{(X,t) \mid X \in \mathcal{A}_t \text{ and } t \in I\}$, we can then identify the family $(g_t \mid t \in I)$ with the mapping $g : \mathcal{M} \longrightarrow \mathcal{S}$ defined by

$$g(X,t) := g_t(X) \quad \text{for all} \quad X \in \mathcal{A}_t \text{ and } t \in I\ . \tag{5.3}$$

Assume now that the terms in the family $(\mathcal{A}_t \mid t \in I)$ are all equal to a fixed set $\mathcal{A}$, so that $\mathcal{M} = \mathcal{A} \times I$ and that $\mathcal{S}$ is a Euclidean space or linear space. Then it makes sense to consider the case when $g(X,\cdot)$ is of class C^1 for all $X \in \mathcal{A}$ and consider the time-family $(g_t^\bullet \mid t \in I)$ of mappings defined by

$$g_t^\bullet(X) := g(X,\cdot)^\bullet(t) \qquad \text{for all} \quad X \in \mathcal{A} \text{ and } t \in I\ , \tag{5.4}$$

which we call the time-derivative of the time-family $(g_t \mid t \in I)$. The formula (5.4) can also be applied when $\mathcal{M}$ is a subset of $\mathcal{A} \times I$ such that, for each $(X,t) \in \mathcal{M}$, there is a neighborhood of t such that $(X,s) \in \mathcal{M}$ for all s in this neighborhood.

In general, all equations, involving either mappings or families, are understood to hold value-wise or term-wise.

6. Motions

We assume that a time-interval I and a fixed frame-space $\mathcal{F}$ with translation space $\mathcal{V}$ are given.

Definition 5. *A* **motion** *is a C^2 mapping $\bar{\mu} : \mathcal{B} \times I \longrightarrow \mathcal{F}$ such that for each $t \in I$, $\bar{\mu}_t := \bar{\mu}(\cdot,t) \in \mathrm{Pl}\,\mathcal{B}$. Thus, a motion can also be viewed as a time-family of placements in the space $\mathcal{F}$. The* **trajectory** *of the motion $\bar{\mu}$ is the set*

$$\mathcal{M} := \{(\bar{\mu}_t(X),t) \mid (X,t) \in \mathcal{B} \times I\} \subset \mathcal{F} \times I\ . \tag{6.1}$$

A mapping from $\mathcal{B} \times I$ to some linear space will be called a **material field***, and a mapping from the trajectory $\mathcal{M}$ to some linear space will be called a* **spatial field**.[5]

We assume now that a motion $\bar{\mu}$ as just defined is given. A material field can be used to generate a spatial field and vice vera in the following way. Let $\mathcal{W}$ be a linear space and $\Phi : \mathcal{B} \times I \longrightarrow \mathcal{W}$ be a material field. We can define the **associated spatial field** $\Phi_{\mathrm{s}} : \mathcal{M} \longrightarrow \mathcal{W}$ by

$$\Phi_{\mathrm{s}}(x,t) := \Phi(\bar{\mu}_t^{\leftarrow}(x),t) \quad \text{for all} \quad (x,t) \in \mathcal{M}\ . \tag{6.2}$$

Given a spatial field $\Psi : \mathcal{M} \longrightarrow \mathcal{W}$ we can define the **associated material field** $\Psi_{\mathrm{m}} : \mathcal{B} \times I \longrightarrow \mathcal{W}$ by

$$\Psi_{\mathrm{m}}(X,t) := \Psi(\bar{\mu}_t(X),t) \quad \text{for all} \quad (X,t) \in \mathcal{B} \times I\ . \tag{6.3}$$

Note that $(\Phi_{\mathrm{s}})_{\mathrm{m}} = \Phi$ and $(\Psi_{\mathrm{m}})_{\mathrm{s}} = \Psi$.

If a material field is continuous, of class C^1, or class C^2, so is the associated spatial field and vice versa. If they are of class C^1, we use the notations

$$\Phi^\bullet(X,t) := \Phi(X,\cdot)^\bullet(t), \quad \nabla\Phi(X,t) := \nabla(\Phi(\cdot,t))(X) \quad \text{for all} \quad (X,t) \in \mathcal{B} \times I \tag{6.4}$$

and

$$\Psi^\bullet(x,t) := \Psi(x,\cdot)^\bullet(t), \quad \nabla\Psi(x,t) := \nabla(\Psi(\cdot,t))(x) \quad \text{for all} \quad (x,t) \in \mathcal{M}\ . \tag{6.5}$$

Assume that Φ and Ψ are of class C^1. Using (5.3) we can then consider the time-families $(\Phi_t \mid t \in I)$ and $(\Psi_t \mid t \in I)$ and their time-derivatives $(\Phi_t^\bullet \mid t \in I)$ and $(\Psi_t^\bullet \mid t \in I)$. Of course, $\Phi^\bullet$ is a continuous material field and $\Psi^\bullet$ is a continuous spatial field. They are of class C^1 if the original fields were of class C^2.

The **spatial velocity** $\bar{\mathbf{v}} : \mathcal{B} \times I \longrightarrow \mathcal{V}$ and **spatial acceleration** $\bar{\mathbf{a}} : \mathcal{B} \times I \longrightarrow \mathcal{V}$ are defined by

$$\bar{\mathbf{v}} := (\bar{\mu}^\bullet)_{\mathrm{s}}, \quad \text{and} \quad \bar{\mathbf{a}} := (\bar{\mu}^{\bullet\bullet})_{\mathrm{s}}\ . \tag{6.6}$$

[5]In much of the past literature, the terms “Lagrangian” and “Eulerian” have been used instead of “spatial” and “material”. This is unfortunate because these terms are non-descripive and historically inaccurate.

We use the following notation for the velocity gradient and its value-wise symmetric part:

$$\bar{\mathbf{L}} := \nabla\bar{\mathbf{v}} : \mathcal{M} :\longrightarrow \operatorname{Lin}\mathcal{V} \ , \tag{6.7}$$

$$\bar{\mathbf{D}} := \frac{1}{2}(\bar{\mathbf{L}} + \bar{\mathbf{L}}^{\top}) : \mathcal{M} \longrightarrow \operatorname{Sym}\mathcal{V} \ . \tag{6.8}$$

Now let a spatial field $\Psi : \mathcal{M} \longrightarrow \mathcal{W}$ field be given. The **material time-derivateve** Ψ° of the spatial field Ψ is the spatial field defined by

$$\Psi^{\circ} := ((\Psi_{\mathrm{m}})^{\bullet})_{\mathrm{s}} \ . \tag{6.9}$$

Using (6.3) and the chain rule, it follows that

$$\Psi^{\circ} = \Psi^{\bullet} + \nabla\Psi\bar{\mathbf{v}}. \tag{6.10}$$

Applying (6.10) to the case when Ψ is the spatial velocity, we obtain the following relation between the spatial fields associated with the velocity and accelertion:

$$\bar{\mathbf{a}} = \bar{\mathbf{v}}^{\bullet} + \bar{\mathbf{L}}\bar{\mathbf{v}} \ . \tag{6.11}$$

We use the notation

$$\bar{\mathbf{M}}(X,t) := \bar{\mathbf{M}}_t(X) := \nabla\bar{\mu}_t(X) \in \operatorname{Lis}(\mathcal{T}_X, \mathcal{V}) \quad \text{for all} \quad (X,t) \in \mathcal{B} \times I \ , \tag{6.12}$$

and, for each $X \in \mathcal{B}$, we call the mapping $\bar{\mathbf{M}}(X,\cdot) : I \longrightarrow \operatorname{Lis}(\mathcal{T}_X, \mathcal{V})$ the **motion of the body element** $\mathcal{T}_X$ induced by the motion $\bar{\mu}$ of the whole body.

It is sometimes useful to specify a fixed **reference placement** $\kappa : \mathcal{B} \longrightarrow \mathcal{B}_\kappa$ in the frame-space $\mathcal{F}$ and characterize all other placements μ in $\mathcal{F}$ by their transplacements from κ. Thus, we obtain the **transplacement process** $\bar{\chi} : \mathcal{B}_\kappa \times I \longrightarrow \mathcal{F}$ given by

$$\bar{\chi}(p,t) := \bar{\mu}(\kappa^{\leftarrow}(p),t) \quad \text{for all} \quad (p,t) \in \mathcal{B}_\kappa \times I \ . \tag{6.13}$$

We use the notation

$$\mathbf{K}(X) := \nabla_X\kappa \qquad \text{for all} \qquad X \in \mathcal{B} \tag{6.14}$$

and call, for each $X \in \mathcal{B}$, $\mathbf{K}(X)$ the **reference placement of the body element** $\mathcal{T}_X$ induced by the reference placement κ of the whole body. The **transplacement gradient** $\bar{\mathbf{F}} : \mathcal{B}_\kappa \times I \longrightarrow \operatorname{Lis}\mathcal{V}$ is given by

$$\bar{\mathbf{F}}(p,t) = \nabla\bar{\chi}(p,t) = \bar{\mathbf{M}}_t(\kappa^{\leftarrow}(p))\mathbf{K}^{-1}(\kappa^{\leftarrow}(p)) \quad \text{for all} \quad (p,t) \in \mathcal{B}_\kappa \times I \ . \tag{6.15}$$

7. Densities and Contactors

Definition 6. *We say that a part $\mathcal{P} \in \Omega_{\mathcal{B}}$ is* **internal** *if, for every placement $\mu \in \operatorname{Pl}\mathcal{B}$, we have $\operatorname{Clo}\mathcal{P}_\mu \subset \mathcal{B}_\mu$. We denote the set of all internal parts by $\Omega_{\mathcal{B}}^{\mathrm{int}}$.*

It is easily seen, using the properties (B_2) and (T_2) in Section 3, that in this definition "every" can be replaced by "some"' without changing the meaning.

We assume now that a linear space $\mathcal{W}$ is given.

Definition 7. *An additive mapping $\mathbf{H} : \Omega_{\mathcal{B}} \longrightarrow \mathcal{W}$ is said to have* **densities** *if, for every $\mu \in \operatorname{Pl}\mathcal{B}$, there is a continuous mapping $\mathbf{h}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{W}$ such that*

$$\mathbf{H}(\mathcal{P}) = \int_{\mathcal{P}_\mu} \mathbf{h}_\mu \qquad \text{for all} \quad \mathcal{P} \in \Omega_{\mathcal{B}}^{\mathrm{int}} \ . \tag{7.1}$$

We call $\mathbf{h}_\mu$ the **density** *of* $\mathbf{H}$ *in the placement μ.*

Let $\mu, \mu' \in \operatorname{Pl}\mathcal{B}$ be two placements such that (7.1) holds for μ. Consider the transplacement $\alpha := \mu\circ\mu'^{\leftarrow} : \mathcal{B}_{\mu'} \longrightarrow \mathcal{B}_\mu$. By the Theorem on Transformation of Volume Integrals (see Section 410 in [FDS], Volume II) and (4.10) we have

$$\mathbf{H}(\mathcal{P}) = \int_{\mathcal{P}_\mu} \mathbf{h}_\mu = \int_{\mathcal{P}_{\mu'}} \rho_{\mu',\mu}(\mathbf{h}_\mu \circ \alpha) \quad \text{for all} \quad \mathcal{P} \in \Omega_{\mathcal{B}}^{\mathrm{int}} \ . \tag{7.2}$$

Therefore, it follows that in Definition 7, "every" can be replaced by "some" without changing the meaning. Moreover, if $\mathbf{h}_\mu$ in the density of $\mathbf{H}$ in the the placement μ, then $\mathbf{h}_{\mu'} := \rho_{\mu',\mu}(\mathbf{h}_\mu \circ \alpha)$ in the density of $\mathbf{H}$ in the the placement μ'.

Definition 8. *We say that an interaction $\mathbf{I} : (\Omega_{\mathcal{B}})^2_{\text{sep}} \longrightarrow \mathcal{W}$ has* **contactors** *if, for every placement μ, there is a C^1 mapping $\mathbf{C}_\mu : \mathcal{B}_\mu \longrightarrow \operatorname{Lin}(\operatorname{Vfr}\mu, \mathcal{W})$ such that*

$$\mathbf{I}(\mathcal{P},\mathcal{Q}) = \int_{\operatorname{Rct}_\mu(\mathcal{P},\mathcal{Q})} \mathbf{C}_\mu \mathbf{n}_{\mathcal{P}_\mu} \quad \text{for all} \;\; (\mathcal{P},\mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\text{sep}} \;\; \text{with} \;\; \mathcal{P} \in \Omega^{\text{int}}_{\mathcal{B}} \; . \tag{7.3}$$

We call $\mathbf{C}_\mu$ the **contactor** *of $\mathbf{I}$ in the placement μ.*

Let $\mu, \mu' \in \operatorname{Pl}\mathcal{B}$ be two placements and assume that (7.3) holds for μ. Consider again the transplacement $\alpha := \mu \circ \mu'^{\leftarrow} : \mathcal{B}_{\mu'} \longrightarrow \mathcal{B}_\mu$. By the Theorem on Transformation of Surface Integrals (see Chapter 5 [FDS], Volume II) and (4.10) we have

$$\mathbf{I}(\mathcal{P},\mathcal{Q}) = \int_{\operatorname{Rct}_{\mu'}(\mathcal{P},\mathcal{Q})} \rho_{\mu',\mu}(\mathbf{C}_\mu \circ \alpha)(\nabla\alpha)^{-\top} \mathbf{n}_{\mathcal{P}_{\mu'}} \quad \text{for all} \;\; (\mathcal{P},\mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\text{sep}} \;\; \text{with} \;\; \mathcal{P} \in \Omega^{\text{int}}_{\mathcal{B}} \; . \tag{7.4}$$

Since $\mathbf{C}_\mu$ is of class C^1, so is $\mathbf{C}_{\mu'} := \rho_{\mu',\mu}(\mathbf{C}_\mu \circ \alpha)(\nabla\alpha)^{-\top}$ and hence is the contactor of $\mathbf{I}$ in the placement μ'. We conclude, again, that in Definition 8, "every" can be replaced by "some" without changing the meaning.

In the case when $\mathcal{Q} := \mathcal{P}^{\text{rem}}$, (7.3) reduces to

$$\operatorname{Res}_{\mathbf{I}}(\mathcal{P}) = \int_{\operatorname{Rby}\mathcal{P}_\mu} \mathbf{C}_\mu \mathbf{n}_{\mathcal{P}_\mu} \quad \text{for all} \;\; \mathcal{P} \in \Omega^{\text{int}}_{\mathcal{B}} \; . \tag{7.5}$$

Theorem 4: *Given an interaction $\mathbf{I} : (\Omega_{\mathcal{B}})^2_{\text{sep}} \longrightarrow \mathcal{W}$ with contactors and an additive mapping $\mathbf{H} : \Omega_{\mathcal{B}} \longrightarrow \mathcal{W}$ with densities, the following three conditions are equivalent:*

1)We have

$$\operatorname{Res}_{\mathbf{I}}(\mathcal{P}) + \mathbf{H}(\mathcal{P}) = \mathbf{0} \quad \text{for all} \;\; \mathcal{P} \in \Omega^{\text{int}}_{\mathcal{B}} \; . \tag{7.6}$$

2) For every placement $\mu \in \operatorname{Pl}\mathcal{B}$, we have

$$\operatorname{div}\mathbf{C}_\mu + \mathbf{h}_\mu = \mathbf{0} \tag{7.7}$$

where $\mathbf{h}_\mu$ is the density of $\mathbf{H}$ in the placement μ, and $\mathbf{C}_\mu$ is the **contactor** *of $\mathbf{I}$ in the placement μ.*

3) Condition 2) holds with "every" replaced by "some".

Proof: Assume that (7.6) holds, let $\mu \in \operatorname{Pl}\mathcal{B}$ be given, let $\mathbf{h}_\mu$ be the density of $\mathbf{H}$ in the placement μ as characterized by (7.1), and let $\mathbf{C}_\mu$ is the **contactor** of $\mathbf{I}$ in the placement μ as characterized by (7.3). In view of (7.1) and (7.5), (7.6) is equivalent to

$$\int_{\operatorname{Rby}\mathcal{P}_\mu} \mathbf{C}_\mu \mathbf{n}_{\mathcal{P}_\mu} + \int_{\mathcal{P}_\mu} \mathbf{h}_\mu = \mathbf{0} \quad \text{for all} \;\; \mathcal{P} \in \Omega^{\text{int}}_{\mathcal{B}} \; . \tag{7.8}$$

Since $\mathbf{C}_\mu$ is of class C^1 we can use the divergence theorem (3.2) to show that (7.8) is equivalent to

$$\int_{\mathcal{P}_\mu} (\operatorname{div}\mathbf{C}_\mu + \mathbf{h}_\mu) = \mathbf{0} \quad \text{for all} \;\; \mathcal{P} \in \Omega^{\text{int}}_{\mathcal{B}} \; . \tag{7.9}$$

Since $\operatorname{div}\mathbf{C}_\mu$ and $\mathbf{h}_\mu$ are continuous and (7.9) holds for all interior parts we see that (7.9) is equivalent to

$$\operatorname{div}\mathbf{C}_\mu + \mathbf{h}_\mu = \mathbf{0} \; . \tag{7.10}$$

Since $\mu \in \operatorname{Pl}\mathcal{B}$ was arbitrary this implies that condition 2 is valid. If (7.7) is valid just for some $\mu \in \operatorname{Pl}\mathcal{B}$ then the equivalences mentioned show that (7.6) holds. ∎

The proof of the following result is analogous the the proof just presented.

Theorem 5: *Given a real valued interaction $I : (\Omega_{\mathcal{B}})^2_{\text{sep}} \longrightarrow \mathbb{R}$ with contactors and a real valued additive mapping $H : \Omega_{\mathcal{B}} \longrightarrow \mathbb{R}$ with densities, then the following three conditions are equivalent:*

1)We have

$$\mathrm{Res}_I(\mathcal{P}) + H(\mathcal{P}) \geq 0 \quad \text{for all} \quad \mathcal{P} \in \Omega_{\mathcal{B}}^{\text{int}} \ , \tag{7.11}$$

2) For every placement $\mu \in \mathrm{Pl}\,\mathcal{B}$, we have

$$\mathrm{div}\,\mathbf{c}_\mu + h_\mu \geq 0 \ . \tag{7.12}$$

where h_μ is the density of H in the placement μ, and $\mathbf{c}_\mu$ is the **contactor** *of I in the placement μ.*

3) Condition 2) hold with "every" replaced by "some".

Remark 4: One can modify Definition 8 by assuming that the values of $\mathbf{C}_\mu$ are merely continuous mappings rather then linear mappings. In this case $\mathbf{C}_\mu$ should be called a *proto-contactor*. Then, if the interaction is equal to a mapping with densities, one can prove that the values of $\mathbf{C}_\mu$ must be linear and hence that $\mathbf{C}_\mu$ is actually a contactor. The first proof was essentially given by Cauchy in 1823. Later, in 1958, Noll proved that the balance law can even be used to prove the existence of a proto-contactor with a suitable definition of a surface interaction and suitable regularity assumptions. A detailed discussion of this issue is given in [N3]. ▮

It is often useful to introduce a **reference mass**, which is an additive function $m : \Omega_{\mathcal{B}} \longrightarrow \mathbb{P}^\times$ with densities. Given a placement μ and a sub-body $\mathcal{P} \in \Omega_{\mathcal{B}}$ the **mass** of this sub-body is given by

$$m(\mathcal{P}) = \int_{\mathcal{P}_\mu} \rho_\mu \tag{7.13}$$

where $\rho_\mu : \mathcal{B}_\mu \longrightarrow \mathbb{P}^\times$ is the density of m in the placement μ and is called the **mass-density** of the body in the placement μ. It follows from (7.2) that if μ' is another placement

$$\rho_\mu \circ \mu = \rho_{\mu'} \circ \mu' \quad \text{when} \quad \delta_\mu = \delta_{\mu'} \ . \tag{7.14}$$

From now on we assume that a reference mass is given.

Theorem 6: *Let $\mathbf{H} : \Omega_{\mathcal{B}} \longrightarrow \mathcal{W}$ be an additive mapping with densities. Then there is a mapping $\mathbf{h} : \mathcal{B} \longrightarrow \mathcal{W}$ of $\mathbf{H}$, called the* **specific density** *of $\mathbf{H}$, such that*

$$\mathbf{h} = \frac{\mathbf{h}_\mu}{\rho_\mu} \circ \mu \qquad \text{for all} \ \ \mu \in \mathrm{Pl}\,\mathcal{B} \ . \tag{7.15}$$

Proof: Let $\mu, \mu' \in \mathrm{Pl}\,\mathcal{B}$ be given. Using (7.2) we have $\mathbf{h}_{\mu'} = \rho_{\mu',\mu}(\mathbf{h}_\mu \circ \mu \circ \mu'^{\leftarrow})$ and $\rho_{\mu'} = \rho_{\mu',\mu}(\rho_\mu \circ \mu \circ \mu'^{\leftarrow})$. It follows that

$$\frac{\mathbf{h}_{\mu'}}{\rho_{\mu'}} \circ \mu' = \frac{\mathbf{h}_\mu \circ \mu \circ \mu'^{\leftarrow}}{\rho_\mu \circ \mu \circ \mu'^{\leftarrow}} \circ \mu' = \frac{\mathbf{h}_\mu}{\rho_\mu} \circ \mu \ . \tag{7.16}$$

Since the placements μ and μ' were arbitrary, this proves the theorem. ▮

In light of Theorem 6 we will use the following notation:

$$\int_{\mathcal{P}} \mathbf{h}\,\mathrm{d}m := \mathbf{H}(\mathcal{P}) = \int_{\mathcal{P}_\mu} \mathbf{h}_\mu = \int_{\mathcal{P}_\mu} \rho_\mu(\mathbf{h} \circ \mu^{\leftarrow}) \quad \text{for all} \ \ \mu \in \mathrm{Pl}\,\mathcal{B} \text{ and } \mathcal{P} \in \Omega_{\mathcal{B}}^{\text{int}} \ . \tag{7.17}$$

Let $\bar{\mathbf{H}} := (\bar{\mathbf{H}}_t : \Omega_{\mathcal{B}} \longrightarrow \mathcal{W} \mid t \in I)$ be a time-family of additive mappings. As explained in Section 5, this time-family can be identified with a mapping $\bar{\mathbf{H}} : \Omega_{\mathcal{B}} \times I \longrightarrow \mathcal{W}$. We say that $\bar{\mathbf{H}}$ is of class C^1 if the mapping $\bar{\mathbf{H}}(\mathcal{P}, \cdot) : I \longrightarrow \mathcal{W}$ is of class C^1 for all $\mathcal{P} \in \Omega_{\mathcal{B}}$. If this is the case, we can form the time-derivative $\bar{\mathbf{H}}^\bullet_t(\mathcal{P}) := (\bar{\mathbf{H}}(\mathcal{P}, \cdot))^\bullet(t)$ for all $\mathcal{P} \in \Omega_{\mathcal{B}}$ and $t \in I$ of the given family. It is clear that this time-derivative is also a time-family of additive mappings.

We now assume that each mapping in the time-family $\bar{\mathbf{H}}$ has densities in the sense of Definition 7. Let $\bar{\mathbf{h}} : \mathcal{B} \times I \longrightarrow \mathcal{W}$ be the mapping such that $\bar{\mathbf{h}}_t := \bar{\mathbf{h}}(\cdot, t)$ is the specific density of $\bar{\mathbf{H}}_t$ for all $t \in I$. If $\bar{\mathbf{h}}$ is of class C^1 then so is $\bar{\mathbf{H}}$ and we have

$$\bar{\mathbf{H}}^\bullet_t(\mathcal{P}) = \int_{\mathcal{P}} \bar{\mathbf{h}}^\bullet_t \mathrm{d}m = \int_{\mathcal{P}_\mu} \rho_\mu(\bar{\mathbf{h}}^\bullet_t \circ \mu^{\leftarrow}) \quad \text{for all} \ \ t \in I, \ \mu \in \mathrm{Pl}\mathcal{B} \text{ and } \mathcal{P} \in \Omega_{\mathcal{B}} \ . \tag{7.18}$$

Remark 5: In practice the reference mass is usually taken to be the inertial-gravitational mass. However, here we do not assume that this is the case. There could be situations in which it is useful to take the referential mass to be different from the inertial-gravitational mass. For example, one may wish to fix a reference configuration δ_R and define,, $m(\mathcal{P})$ to be the volume of the region $\mathrm{imb}_{\delta_R>}(\mathcal{P})$ for every $\mathcal{P} \in \Omega_{\mathcal{B}}$.

In (7.17) the reference mass is, in the language of measure theory, a *measure* on $\mathcal{B}$. Given an additive mapping with density $\mathbf{H}$, the mapping $\mathbf{h}$ in (7.15) is nothing but the density of $\mathbf{H}$ with respect to the measure m. ▮

8. Balance of Forces and Torques

It is often useful to fix a frame-space $\mathcal{F}$, with translation space $\mathcal{V}$, and confine one's attention to placements whose range-space is $\mathcal{F}$. It is then useful to consider force systems with values in $\mathcal{V}$, independent of the choice of a configuration, as follows:

Definition 9. *A* **force system** *in the space $\mathcal{V}$ is a pair $(\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}, \mathbf{F}^{\mathrm{e}}_{\mathcal{V}})$, where $\mathbf{F}^{\mathrm{i}}_{\mathcal{V}} : (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \longrightarrow \mathcal{V}$ is an interaction and $\mathbf{F}^{\mathrm{e}}_{\mathcal{V}} : \Omega_{\mathcal{B}} \longrightarrow \mathcal{V}$ is additive. The mapping $\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}$ is called the* **internal force system** *in $\mathcal{V}$ and $\mathbf{F}^{\mathrm{e}}_{\mathcal{V}}$ is called the* **external force system** *in $\mathcal{V}$.*

Let a force system $(\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}, \mathbf{F}^{\mathrm{e}}_{\mathcal{V}})$ in $\mathcal{V}$ be a given. The first fundamental law of mechanics, called the **Balance of Forces**, says:

$$\mathrm{Res}_{\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}}(\mathcal{P}) + \mathbf{F}^{\mathrm{e}}_{\mathcal{V}}(\mathcal{P}) = \mathbf{0} \quad \text{for all} \;\; \mathcal{P} \in \Omega_{\mathcal{B}} \; . \tag{8.1}$$

We say that the system $(\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}, \mathbf{F}^{\mathrm{e}}_{\mathcal{V}})$ is **force-balanced** if (8.1) holds.

Since $\mathbf{F}^{\mathrm{e}}_{\mathcal{V}}$ is additive, the following **Law of Action and Reaction** is an immediate consequence of (8.1) and Theorem 3: The internal force system is skew, i.e.,

$$\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}(\mathcal{P}, \mathcal{Q}) = -\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}(\mathcal{Q}, \mathcal{P}) \qquad \text{for all} \quad (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \; . \tag{8.2}$$

Remark 6: The law of action and reaction is often referred to as *Newton's Third Law.* Thus, if one assumes the balance law (8.1), one can prove Newton's Third Law instead of assuming it a priori, as Newton and many physics textbooks since Newton have done. The balance of forces has been understood by engineers, if only implicitly, since antiquity. ▮

We assume now that $\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}$ has contactors and $\mathbf{F}^{\mathrm{e}}_{\mathcal{V}}$ has densities.

Let μ be a placement of the body in $\mathcal{F}$ and put $\mathcal{B}_\mu := \mu_>(\mathcal{B})$. Let $\mathbf{T}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Lin}\,\mathcal{V}$ denote the contactor for $\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}$ and let $\mathbf{b}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ denote the density of $\mathbf{F}^{\mathrm{e}}_{\mathcal{V}}$ in the placement μ. It follows from Theorem 4 that (8.1), restricted to internal parts $\mathcal{P}$, is equivalent to

$$\mathrm{div}\mathbf{T}_\mu + \mathbf{b}_\mu = \mathbf{0} \; . \tag{8.3}$$

Definition 10. *A* **torque system** *in the space $\mathcal{V}$ is a pair $(\mathbf{M}^{\mathrm{i}}_{\mathcal{V}}, \mathbf{M}^{\mathrm{e}}_{\mathcal{V}})$, where $\mathbf{M}^{\mathrm{i}}_{\mathcal{V}} : (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \longrightarrow \mathrm{Skew}\,\mathcal{V}$ is an interaction and $\mathbf{M}^{\mathrm{e}}_{\mathcal{V}} : \Omega_{\mathcal{B}} \longrightarrow \mathrm{Skew}\,\mathcal{V}$ is additive. The mapping $\mathbf{M}^{\mathrm{i}}_{\mathcal{V}}$ is called the* **internal torque system** *and $\mathbf{M}^{\mathrm{e}}_{\mathcal{V}}$ is called the* **external torque system**.

Let $(\mathbf{M}^{\mathrm{i}}_{\mathcal{V}}, \mathbf{M}^{\mathrm{e}}_{\mathcal{V}})$ be a torque system in $\mathcal{V}$. The second fundamental law of mechanics is the **Balance of Torques**, which states that

$$\mathrm{Res}_{\mathbf{M}^{\mathrm{i}}_{\mathcal{V}}}(\mathcal{P}) + \mathbf{M}^{\mathrm{e}}_{\mathcal{V}}(\mathcal{P}) = \mathbf{0} \quad \text{for all} \;\; \mathcal{P} \in \Omega_{\mathcal{B}} \; . \tag{8.4}$$

Again, an immediate consequence of (8.4) and Theorem 3 is the following: The internal torque system is skew, i.e.,

$$\mathbf{M}^{\mathrm{i}}_{\mathcal{V}}(\mathcal{P}, \mathcal{Q}) = -\mathbf{M}^{\mathrm{i}}_{\mathcal{V}}(\mathcal{Q}, \mathcal{P}) \qquad \text{for all} \quad (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \; . \tag{8.5}$$

Remark 7: The balance of torques has also been understood by engineers, if only implicitly, since antiquity. Archimedes' work on levers, which essentially dealt with torques, caused him to remark: "Give me a place to stand on, and I will move the Earth." ▮

Here we will assume that all torques come from forces. When (8.4) is considered only in cases when $\mathcal{P}$ is internal, this means the following: Choosing $q \in \mathcal{E}$ arbitrarily, $\mathbf{M}^{\mathrm{i}}_{\mathcal{V}}$ has a contactor $\mathbf{C}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Lin}\,(\mathcal{V}, \mathrm{Skew}\,\mathcal{V})$ and $\mathbf{M}^{\mathrm{e}}_{\mathcal{V}}$ has a density $\mathbf{m}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ in a given placement μ, and they are given by

$$\begin{array}{l} \mathbf{C}_\mu(x)\mathbf{u} := (x-q) \wedge \mathbf{T}_\mu(x)\mathbf{u} \\ \mathbf{m}_\mu(x) := (x-q) \wedge \mathbf{b}_\mu(x) \end{array} \quad \text{for all } \ \mathbf{u} \in \mathcal{V} \text{ and } x \in \mathcal{B}_\mu \ . \tag{8.6}$$

A calculation using the results from Chapter 6 of [FDS], shows that the divergence of $\mathbf{C}_\mu$ is given by

$$\mathrm{div}_x \mathbf{C}_\mu = \mathbf{T}^\top_\mu(x) - \mathbf{T}_\mu(x) + (x-q) \wedge \mathrm{div}_x \mathbf{T}_\mu \quad \text{for all } \ x \in \mathcal{B} \ . \tag{8.7}$$

Using Theorem 4, it follows from the balance law (8.4), restricted to internal parts, that $\mathrm{div}\mathbf{C}_\mu + \mathbf{m}_\mu = \mathbf{0}$. Combining this result with (8.7) and (8.6), we obtain

$$\mathbf{T}^\top_\mu(x) - \mathbf{T}_\mu(x) + (x-q) \wedge (\mathrm{div}_x \mathbf{T}_\mu + \mathbf{b}_\mu(x)) = \mathbf{0} \quad \text{for all } \ x \in \mathcal{B}_\mu \ . \tag{8.8}$$

Using (8.3) we find that the condition

$$\mathrm{Rng}\,\mathbf{T}_\mu \subset \mathrm{Sym}\ \mathcal{V} \tag{8.9}$$

is equivalent to balance of torques for internal parts.

We say that the system $(\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}, \mathbf{F}^{\mathrm{e}}_{\mathcal{V}})$ is **torque-balanced** if the system of torques derived from it satisfies (8.4).

Remark 8: When $\mathcal{P}$ is not internal, (8.1) and (8.4) must be taken into account when considering what are often called *boundary conditions.* ∎

From here on we will assume that (8.3) and (8.9) are valid. We adjust the codomain of $\mathbf{T}_\mu$ to Sym $\mathcal{V}$ without change of notation and call $\mathbf{T}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Sym}\,\mathcal{V}$ the **Cauchy stress** of the force system in the placement μ and the mapping $\mathbf{b}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ the **external body force** in the placement μ.

It is sometimes useful to specify a fixed **reference placement** $\kappa : \mathcal{B} \longrightarrow \mathcal{B}_\kappa$ in the frame space $\mathcal{F}$ and characterize all other placements μ in $\mathcal{F}$ by their transplacements

$$\chi := \mu \circ \kappa^{\leftarrow} : \mathcal{B}_\kappa \longrightarrow \mathcal{B}_\mu, \tag{8.10}$$

as in Section 6, from the reference placement. The **transplacement gradient** $\mathbf{F} : \mathcal{B}_\kappa \longrightarrow \mathrm{Lin}\,\mathcal{V}$, defined by $\mathbf{F} := \nabla\chi$, can then be used to represent an internal force interaction whose contactor in the placement μ is the Cauchy stress $\mathbf{T}_\mu$, by a contactor in the reference placement κ. Using (7.4), we see that this contactor is given by

$$\mathbf{T}_{\mathrm{R}}(p) := |\mathrm{det}\mathbf{F}(p)|\mathbf{T}_\mu(\chi(p))\mathbf{F}^{-\top}(p) \quad \text{for all } \ p \in \mathcal{B}_\kappa \ . \tag{8.11}$$

$\mathbf{T}_{\mathrm{R}} : \mathcal{B}_\kappa \longrightarrow \mathrm{Lin}\,\mathcal{V}$ is usually called the **Piola-Kirchhoff stress** (see (43 A.3) of [NLFT]). Note that $\mathbf{T}_{\mathrm{R}}$ does not have symmetric values. Instead, since $\mathbf{T}_\mu$ has symmetric values, it follows from (8.11) that $\mathbf{T}_{\mathrm{R}}$ must satisfy

$$\mathbf{T}_{\mathrm{R}}\mathbf{F}^\top = \mathbf{F}\mathbf{T}^\top_{\mathrm{R}} \ . \tag{8.12}$$

The transplacement gradient can also be used to represent the external force system whose density in the placement μ is the external body force $\mathbf{b}_\mu$, by a density in the reference placement κ. Using (7.2), we see that this density is given by

$$\mathbf{b}_{\mathrm{R}}(p) := |\mathrm{det}\mathbf{F}(p)|\mathbf{b}_\mu(\chi(p)) \qquad \text{for all} \qquad p \in \mathcal{B}_\kappa \ . \tag{8.13}$$

$\mathbf{b}_{\mathrm{R}} : \mathcal{B}_\kappa \longrightarrow \mathcal{V}$ may be called the **external referential body force** for the placement μ.

Remark 9: The description of force systems described so far is equivalent to the one given in the traditional textbooks, for example in [NLFT] or [G]. It has the disadvantage that it involves an external frame-space $\mathcal{F}$, often considered to be an *absolute* space. In Part 1 of [FC], it is shown that such a space is an illusion and why this illusion is widespread.

The *principle of frame-indifference* states that constitutive laws should not depend on whatever external frame of reference is used to describe them. It will be vacuously satisfied if no external frames of reference are used to state these laws. Therefore, it is useful to describe force systems without using an external frame-space, which we will do below. ∎

Let a configuration $\delta \in \operatorname{Conf}\mathcal{B}$ be given. As in Section 3, we denote the imbedding space for δ by $\mathcal{E}_\delta$ and its translation space by $\mathcal{V}_\delta$, and we use the results of Section 7 in the case when $\mu := \mathrm{imb}_\delta$ and write, for simplicity, δ rather than imb_δ as a subscript.

Definition 11. *A* **force system in the configuration** δ *is a pair* $(\mathbf{F}^{\mathrm{i}}_\delta, \mathbf{F}^{\mathrm{e}}_\delta)$ *which is a force system in the space* $\mathcal{V}_\delta$ *in the sense of Definition 9.*

Let such a force system $(\mathbf{F}^{\mathrm{i}}_\delta, \mathbf{F}^{\mathrm{e}}_\delta)$ in $\mathcal{V}_\delta$ be given. We assume that the balance of forces and the balance of torques are valid, that $\mathbf{F}^{\mathrm{i}}_\delta$ has contactors and that $\mathbf{F}^{\mathrm{e}}_\delta$ has densities. The results (8.3) and (8.9) remain valid when the subscript δ is used instead of μ, when $\mathbf{T}_\delta$ is interpreted to be the contactor of $\mathbf{F}^{\mathrm{i}}_\delta$ in the placement imb_δ, and when $\mathbf{b}_\delta$ is interpreted to be the density of $\mathbf{F}^{\mathrm{e}}_\delta$ in the placement imb_δ. We may call $\mathbf{T}_\delta$ the **configurational stress** and imb_δ the **external configurational body force** for δ.

Since $\mathbf{I}_\delta(X)$, defined in (3.5), is a linear isomorphism from $\mathcal{T}_X$ to $\mathcal{V}_\delta$, it can be used to transform the mappings $\mathbf{T}_\delta$ and $\mathbf{b}_\delta$, whose codomains involve $\mathcal{V}_\delta$, into mappings whose codomains involve $\mathcal{T}_X$. Thus, we define, for every $X \in \mathcal{B}$, the **intrinsic stress** $\mathbf{S}_\delta$ and the **external intrinsic body force** $\mathbf{d}_\delta$ associated with the configuration δ by

$$\mathbf{S}_\delta(X) := \mathbf{I}_\delta^{-1}(X)\mathbf{T}_\delta(\mathrm{imb}_\delta(X))\mathbf{I}_\delta^{-\top}(X) \in \mathrm{Sym}(\mathcal{T}_X^*, \mathcal{T}_X)\ , \tag{8.14}$$

and

$$\mathbf{d}_\delta(X) := \mathbf{I}_\delta^{-1}(X)\mathbf{b}_\delta(\mathrm{imb}_\delta(X)) \in \mathcal{T}_X \quad \text{for all} \quad X \in \mathcal{B}\ . \tag{8.15}$$

respectively.

Remark 10: The mappings $\mathbf{S}_\delta := (X \mapsto \mathbf{S}_\delta(X))$ and $\mathbf{d}_\delta := (X \mapsto \mathbf{d}_\delta(X))$ are cross-sections of fiber bundles. ▮

Let μ be a placement of the body in a fixed frame-space $\mathcal{F}$ as considered in the beginning of this section and put $\mathcal{B}_\mu := \mu_>(\mathcal{B})$. Denote by δ the configuration induced by μ in accord with (4.1). We use $\mathbf{Q}$, as defined by (4.3) to transport the values of a the force system in the configulation δ to $\mathcal{V}$ and obtain a force system in the sense of Definition 9. Then the balance laws (8.1) and (8.4) hold if and only if corresponding balance laws hold for the force system in the configuration δ. The corresponding Cauchy stress $\mathbf{T}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Sym}\,\mathcal{V}$ and the corresponding external body force $\mathbf{b}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ are related to $\mathbf{T}_\delta$ and $\mathbf{b}_\delta$ by

$$\mathbf{T}_\delta \circ \mathrm{imb}_\delta = \mathbf{Q}\mathbf{T}_\mu\mathbf{Q}^\top \circ \mu \ \text{ and } \ \mathbf{b}_\delta \circ \mathrm{imb}_\delta = \mathbf{Q}\mathbf{b}_\mu \circ \mu \ \ , \tag{8.16}$$

respectively. Using (4.4), (4.5), (8.14) and (8.15) we find that the Cauchy stress $\mathbf{T}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Sym}\,\mathcal{V}$ and the external body force $\mathbf{b}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ are related to the intrisic stress and the external intrinsic body force by

$$\mathbf{T}_\mu \circ \mu = \mathbf{M}_\mu \mathbf{S}_\delta \mathbf{M}_\mu^\top \tag{8.17}$$

and

$$\mathbf{b}_\mu \circ \mu = \mathbf{M}_\mu \mathbf{d}_\delta\ . \tag{8.18}$$

9. Deformation Processes and Mechanical Processes

As before, we assume that a time-interval I is given.

Definition 12. *We say that a time-family* $(\bar{\delta}_t \mid t \in I)$ *of configurations, and the corresponding mapping* $\bar{\delta} : \mathcal{B} \times I \longrightarrow \mathrm{Conf}\mathcal{B}$, *is a* **deformation process**.

Let a deformation process $(\bar{\delta}_t \mid t \in I)$, be given. Since $\bar{\delta}_t$ is a configuration it can be used to construct an imbedding space $\mathcal{E}_t := \mathcal{E}_{\bar{\delta}_t}$[6], with translation space $\mathcal{V}_t$, in which $\mathcal{B}$ is imbedded using the mapping $\mathrm{imb}_t := \mathrm{imb}_{\bar{\delta}(t)} : \mathcal{B} \longrightarrow \mathcal{B}_t := \mathcal{B}_{\bar{\delta}(t)}$. We will denote the mappings defined in (3.5) and (3.6) for the imbedding imb_t by $\bar{\mathbf{I}}_t$ and $\bar{\mathbf{G}}_t$, respectively, i.e.,

$$\bar{\mathbf{I}}_t(X) := \nabla_X \mathrm{imb}_t \in \mathrm{Lis}\,(\mathcal{T}_X, \mathcal{V}_t)\ , \quad \bar{\mathbf{G}}_t(X) := \bar{\mathbf{I}}_t^\top(X)\bar{\mathbf{I}}_t(X) \in \mathrm{Pos}^+(\mathcal{T}_X, \mathcal{T}_X^*)\ , \tag{9.1}$$

[6]The time-family $(\mathcal{E}_t \mid t \in I)$ of Euclidean spaces could be interpreted as describing what has been called a neo-classical event world in [N2].

The family $\bar{\mathbf{G}} = (\bar{\mathbf{G}}_t(X) \mid t \in I)$ is called the **deformation process of the body element** $\mathcal{T}_X$ induced by the deformation process $\bar{\delta}$ of the whole body. We say that the deformation process $\bar{\delta}$ is of class C^1 or of class C^2 if $t \mapsto \bar{\mathbf{G}}_t(X)$ is of class C^1 or of class C^2 for all $X \in \mathcal{B}$, respectively. In this case, we define the family $(\bar{\mathbf{G}}^\bullet_t \mid t \in I)$ or the family $(\bar{\mathbf{G}}^{\bullet\bullet}_t \mid t \in I)$ by

$$\bar{\mathbf{G}}^\bullet_t(X) := (s \mapsto \bar{\mathbf{G}}_s(X))^\bullet(t) \quad \text{or} \quad \bar{\mathbf{G}}^{\bullet\bullet}_t(X) := (s \mapsto \bar{\mathbf{G}}_s(X))^{\bullet\bullet}(t) \quad \text{for all} \quad X \in \mathcal{B} \ , \tag{9.2}$$

respectively.

We now let a motion $\bar{\mu}$ as defined by Definition 5 of Section 6 be given. We consider the deformation process $\bar{\delta}$ induced by this motion in the sense that, for each $t \in I$, the placement $\bar{\mu}_t$ induces the configuration $\bar{\delta}_t$ as explained in (4.1). Since the placements imb_t and $\bar{\mu}_t$ both induce the same configuration $\bar{\delta}_t$, it follows from (4.6) that

$$\bar{\mathbf{G}} = \bar{\mathbf{M}}^\top \bar{\mathbf{M}} \ . \tag{9.3}$$

It follows from (4.4), (6.7), (6.3) and the chain rule that

$$\bar{\mathbf{M}}^\bullet = \bar{\mathbf{L}}_{\mathrm{m}} \bar{\mathbf{M}} \ . \tag{9.4}$$

Differentiating (9.3) with respect to time, using (9.4), (6.8), and the product rule, we find

$$\bar{\mathbf{G}}^\bullet = 2\bar{\mathbf{M}}^\top \bar{\mathbf{D}}_{\mathrm{m}} \bar{\mathbf{M}} \ . \tag{9.5}$$

Definition 13. *A* **mechanical process** *is a time-family* $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t) \mid t \in I)$ *of triples where* $(\bar{\delta}_t \mid t \in I)$ *is a deformation process and, for every* $t \in I$, $(\bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t)$ *is a force system in the configuration* $\bar{\delta}_t$*, as defined by Definition 11, which is both force-balanced and torque-balanced.*

Let a mechanical process $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t) \mid t \in I)$, be given. We assume that, for each $t \in I$, $\bar{\mathbf{F}}^{\mathrm{i}}_t$ has contactors, and $\bar{\mathbf{F}}^{\mathrm{e}}_t$ has densities. We can then define, using the formulas (8.14) and (8.15), a time-family $\bar{\mathbf{S}} := (\bar{\mathbf{S}}_t := \mathbf{S}_{\bar{\delta}_t} \mid t \in I)$ of intrinsic stresses and a time-family $\bar{\mathbf{d}} := (\bar{\mathbf{d}}_t := \mathbf{d}_{\bar{\delta}_t} \mid t \in I)$ of external intrinsic body forces for the family $\bar{\delta}$. These two families, together with the time-family $\bar{\mathbf{G}}$, describe the given mechanical process, apart from boundary conditions, without using an external frame of reference.

Now let a mechanical process be given such that its deformation process is the one induced by a given motion $\bar{\mu}$ as defined in Section 6. The considerations of Section 8 then apply for every $t \in I$ with μ and δ replaced by $\bar{\mu}_t$ and $\bar{\delta}_t$. The Cauchy stress and the external body force now become time-families and hence are identified with mappings of the type $\bar{\mathbf{T}} : \mathcal{M} \longrightarrow \mathrm{Sym}\,\mathcal{V}$ and $\bar{\mathbf{b}} : \mathcal{M} \longrightarrow \mathcal{V}$, respectively. By (8.17) and (8.18) the corresponding intrinsic stress $\bar{\mathbf{S}}$ and external intrinsic body force $\bar{\mathbf{d}}$ are related to $\bar{\mathbf{T}}$ and $\bar{\mathbf{b}}$ by

$$\bar{\mathbf{S}} = \bar{\mathbf{M}}^{-1} \bar{\mathbf{T}}_{\mathrm{m}} \bar{\mathbf{M}}^{-\top} \tag{9.6}$$

and

$$\bar{\mathbf{d}} = \bar{\mathbf{M}}^{-1} \bar{\mathbf{b}}_{\mathrm{m}} \ . \tag{9.7}$$

The balance law (8.3) becomes

$$\mathrm{div}\bar{\mathbf{T}} + \bar{\mathbf{b}} = \mathbf{0} \ . \tag{9.8}$$

Using Proposition 2 in Section 67 of [FDS], (6.7), (6.8) and the fact that $\bar{\mathbf{T}}$ has symmetric values, we find that

$$\mathrm{div}(\bar{\mathbf{T}}\bar{\mathbf{v}}) = \mathrm{div}(\bar{\mathbf{T}}) \cdot \bar{\mathbf{v}} + \mathrm{tr}(\bar{\mathbf{T}}\bar{\mathbf{D}}) \ . \tag{9.9}$$

Using (9.9), (9.8) and the divergence theorem (3.2), we conclude that

$$\int_{\mathrm{Rby}\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot \bar{\mathbf{T}}_t \mathbf{n}_{\mathrm{Rby}\mathcal{P}_{\bar{\mu}_t}} + \int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot \bar{\mathbf{b}}_t = \int_{\mathcal{P}_{\bar{\mu}_t}} \mathrm{tr}(\bar{\mathbf{T}}_t \bar{\mathbf{D}}_t) \quad \text{for all} \ \ \mathcal{P} \in \Omega_{\mathcal{B}} \text{ and } t \in I \ . \tag{9.10}$$

The term on the left hand side is the work per unit time, i.e. the power, of the forces acting on the part $\mathcal{P}$.

It easily follows from (9.6) and (9.5) that

$$\frac{1}{2}\mathrm{tr}(\bar{\mathbf{S}}_t \bar{\mathbf{G}}^\bullet_t)(X) = \mathrm{tr}(\bar{\mathbf{T}}_t \bar{\mathbf{D}}_t)(\bar{\mu}_t(X)) \quad \text{for all} \ \ (X, t) \in \mathcal{B} \times I \ , \tag{9.11}$$

which mean that the material field associated with $\mathrm{tr}(\bar{\mathbf{T}}\bar{\mathbf{D}})$, according to (6.3), is

$$\frac{1}{2}\mathrm{tr}(\bar{\mathbf{S}}\bar{\mathbf{G}}^\bullet) = (\mathrm{tr}(\bar{\mathbf{T}}\bar{\mathbf{D}}))_{\mathrm{m}} \ . \tag{9.12}$$

Therefore the **power** of the forces acting on the parts of the body is given by

$$\bar{P}_t(\mathcal{P}) := \int_{\mathcal{P}_{\bar{\mu}_t}} \operatorname{tr}(\bar{\mathbf{T}}_t\bar{\mathbf{D}}_t) = \frac{1}{2}\int_{\mathcal{P}_{\mathrm{imb}_t}} \operatorname{tr}(\bar{\mathbf{S}}_t\bar{\mathbf{G}}_t^{\bullet}) \circ \mathrm{imb}_t^{\leftarrow} \quad \text{for all} \quad \mathcal{P} \in \Omega_{\mathcal{B}} \ , \tag{9.13}$$

and depends only on the mechanical process and not on the motion as seems from the right side of (9.11).

It is clear that the power family $(\bar{P}_t \mid t \in I)$ is a time-family of additive mappings determined by the given mechanical process.

Remark 11: W.N. proved, in 1959, that the balance laws hold if and only if the work done by the forces is frame-indifferent (see [N4]). In view of this fact it is not surprising that the power does not depend on the motion. ▮

10. Energy Balance

Definition 14. *A* **heat transfer system** *is a pair* $(Q^{\mathrm{i}}, Q^{\mathrm{e}})$, *where* $Q^{\mathrm{i}} : (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \longrightarrow \mathbb{R}$ *is an interaction and* $Q^{\mathrm{e}} : \Omega_{\mathcal{B}} \longrightarrow \mathbb{R}$ *is additive. The function* Q^{i} *is called the* **internal heat transfer** *and* Q^{e} *is called the* **external heat absorption**.

Let $(Q^{\mathrm{i}}, Q^{\mathrm{e}})$ be a heat transfer system. We will assume that Q^{i} has contactors and Q^{e} has densities. Let a placement μ be given. Let us denote the contactor of Q^{i} in this placement by[7]

$$-\mathbf{q}_\mu : \mathcal{B}_\mu \longrightarrow \operatorname{Lin}(\operatorname{Vfr}\mu, \mathbb{R}) \cong \operatorname{Vfr}\mu \ . \tag{10.1}$$

The mapping $\mathbf{q}_\mu$ is called the **heat flux** in the placement μ.

Let δ be the configuration induced by μ, as in (4.1), and let $-\mathbf{q}_\delta : \mathcal{B}_\delta \longrightarrow \mathcal{V}_\delta$ denote the contactor of Q^{i} in the placement imb_δ. Since $\mathbf{I}_\delta(X)$, defined in (3.5), is a linear isomorphism from $\mathcal{T}_X$ to $\mathcal{V}_\delta$, it can be used to transform the mapping $\mathbf{q}_\delta$, whose codomain involves $\mathcal{V}_\delta$, into a mapping whose codomain involves $\mathcal{T}_X$. Thus, we define, for every $X \in \mathcal{B}$, the **intrinsic heat flux** $\mathbf{h}_\delta$ associated with the configuration δ by

$$\mathbf{h}_\delta(X) := \mathbf{I}_\delta^{-1}(X)\mathbf{q}_\delta(\mathrm{imb}_\delta(X)) \in \mathcal{T}_X \quad \text{for all} \quad X \in \mathcal{B} \ . \tag{10.2}$$

Remark 12: The mapping $\mathbf{h}_\delta := (X \mapsto \mathbf{h}_\delta(X))$ is a tangent-vector field on $\mathcal{B}$. ▮

As was pointed out in Section 4,

$$\alpha := \mathrm{imb}_\delta \circ \mu^{\leftarrow} : \mathcal{B}_\mu \longrightarrow \mathcal{B}_\delta \tag{10.3}$$

is an adjusted Euclidean isomorphism. Its (constant) gradient $\mathbf{Q} := \nabla\alpha \in \operatorname{Orth}(\operatorname{Vfr}\mu, \mathcal{V}_\delta)$ is an inner-prouct preserving isomorphism. We use $\mathbf{Q}$ to transport the values of a the heat flux $\mathbf{q}_\delta$ in the configulation δ to $\operatorname{Vfr}\mu$. The corresponding heat flux $\mathbf{q}_\mu : \mathcal{B}_\mu \longrightarrow \operatorname{Vfr}\mu$ is related to $\mathbf{q}_\delta$ by

$$\mathbf{q}_\delta \circ \alpha = \mathbf{Q}\mathbf{q}_\mu. \tag{10.4}$$

Using the definition (4.4) of $\mathbf{M}_\mu$, it follows from (10.2) and (4.5) that the heat flux $\mathbf{q}_\mu$ is related to the intrinsic heat flux $\mathbf{h}_\delta$ by

$$\mathbf{q}_\mu \circ \mu = \mathbf{M}_\mu\mathbf{h}_\delta \ . \tag{10.5}$$

Definition 15. *An* **energetical process** *is a sextuplet* $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t, \bar{Q}^{\mathrm{i}}_t, \bar{Q}^{\mathrm{e}}_t, \bar{E}_t) \mid t \in I)$ *of time-families such that* $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t) \mid t \in I)$ *is a mechanical process,* $((\bar{Q}^{\mathrm{i}}_t, \bar{Q}^{\mathrm{e}}_t) \mid t \in I)$ *is a time-family of heat transfer systems, and* $(\bar{E}_t \mid t \in I)$ *is a differentiable time-family of additive mappings, called the* **internal energy**.

We say that a given energetical process $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t, \bar{Q}^{\mathrm{i}}_t, \bar{Q}^{\mathrm{e}}_t, \bar{E}_t) \mid t \in I)$ is **energy-balanced** if

$$\bar{E}^{\bullet}_t(\mathcal{P}) = \bar{P}_t(\mathcal{P}) + \operatorname{Res}_{\bar{Q}^{\mathrm{i}}_t}(\mathcal{P}) + \bar{Q}^{\mathrm{e}}_t(\mathcal{P}) \quad \text{for all} \quad \mathcal{P} \in \Omega_{\mathcal{B}} \text{ and } t \in I \ , \tag{10.6}$$

[7]Here we use the convention that when $\mathbf{q}_\mu$ is pointing away from the body, the body is losing heat. This is the reason for the minus sign.

where $(\bar{P}_t \mid t \in I)$, defined in (9.13), is the power-family determined by the mechanical process $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t) \mid t \in I)$. The formula (10.6) is also known as the *First Law of Thermodynamics*,

From now on we assume that, for all $t \in I$, $\bar{\mathbf{F}}^{\mathrm{e}}_t$, $\bar{Q}^{\mathrm{e}}_t$, $\bar{E}_t$ have densities and that $\bar{\mathbf{F}}^{\mathrm{i}}_t$ and $\bar{Q}^{\mathrm{i}}_t$ have contactors. Also, we assume that a reference mass $m : \Omega_{\mathcal{B}} \longrightarrow \mathbb{P}^{\times}$ as described in Section 7 is given. We define the **specific internal energy** $\bar{\epsilon} : \mathcal{B} \times I \longrightarrow \mathbb{R}$ by the condition that $\bar{\epsilon}_t$ is the specific density of $\bar{E}_t$ for all $t \in I$. We assume that $\bar{\epsilon}$ is of class C^1. The **specific external heat absorption** $\bar{r} : \mathcal{B} \times I \longrightarrow \mathbb{R}$ is defined by the condition that $\bar{r}_t$ is the specific density of $\bar{Q}^{\mathrm{e}}_t$ for all $t \in I$.

Let a frame-space $\mathcal{F}$, with translation space $\mathcal{V}$, be given. Let $\bar{\mu}$ be a motion in $\mathcal{F}$ such that the placement $\bar{\mu}_t$ induces the configuration $\bar{\delta}_t$ for all $t \in I$. We define the **mass-density field** $\bar{\rho} : \mathcal{M} \longrightarrow \mathbb{P}^{\times}$ by $\bar{\rho}_t := \rho_{\bar{\mu}_t}$, the mass-density of the body at time t as characterized by (7.13). Of course $\bar{r}$, $\bar{\epsilon}$ and $\bar{\epsilon}^{\bullet}$ are material fields, as described in Definition 5. Using the associated spatial fields as defined by (6.2) and also (7.18) and (7.17), we see that

$$\bar{E}^{\bullet}_t(\mathcal{P}) = \textstyle\int_{\mathcal{P}} (\bar{\epsilon}^{\bullet})_t \mathrm{d}m = \int_{\mathcal{P}_{\bar{\mu}_t}} ((\bar{\epsilon}^{\bullet})_{\mathrm{s}})_t \bar{\rho}_t \quad , \quad \bar{Q}^{\mathrm{e}}_t(\mathcal{P}) = \int_{\mathcal{P}} \bar{r}_t \mathrm{d}m = \int_{\mathcal{P}_{\bar{\mu}_t}} (\bar{r}_{\mathrm{s}})_t \bar{\rho}_t \qquad \text{for all } \ t \in I \text{ and } \mathcal{P} \in \Omega_{\mathcal{B}} \ . \tag{10.7}$$

Let $\bar{\mathbf{q}}_t$ denote the heat flux in the placement $\bar{\mu}_t$. It can be identified with a spatial field $\bar{\mathbf{q}} : \mathcal{M} \longrightarrow \mathcal{V}$. Using (9.13), and (10.7), we can apply Theorem 4 to conclude that (10.6), limited to internal parts, is equivalent to

$$\bar{\rho}(\bar{\epsilon}^{\bullet})_{\mathrm{s}} = \mathrm{tr}(\bar{\mathbf{T}}\bar{\mathbf{D}}) - \mathrm{div}\bar{\mathbf{q}} + \bar{\rho}\bar{r}_{\mathrm{s}} \ . \tag{10.8}$$

Remark 13: Since $\bar{E}^{\bullet}_t$, $\bar{P}_t$ and $\bar{Q}^{\mathrm{e}}_t$ are all additive, it follows from (10.6) that $\mathrm{Res}_{\bar{Q}^{\mathrm{i}}_t}$ is additive. Assume that $\bar{E}^{\bullet}_t$, $\bar{P}_t$ and $\bar{Q}^{\mathrm{e}}_t$ all have densities. As pointed out in Remark 4, one can then assume that the interactions $\bar{Q}^{\mathrm{i}}_t$ have only proto-contactors and then *prove* that they must have contactors, which means the time-family of internal heat transfer is determined by heat flux vector-fields. ▮

11. Temperature and Entropy

Theorem 7: *Let a heat transfer system* $(Q^{\mathrm{i}}\ , Q^{\mathrm{e}})$*, as defined by Definition 14, and a function* $\theta : \mathcal{B} \longrightarrow \mathbb{P}^{\times}$ *of class* C^1*, called the (absolute)* **temperature***, be given, and assume that* Q^{i} *has contactors and* Q^{e} *has densities. Then there is a pair* $(H^{\mathrm{i}}, H^{\mathrm{e}})$*, where* $H^{\mathrm{i}} : (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \longrightarrow \mathbb{R}$ *is an interaction and* $H^{\mathrm{e}} : \Omega_{\mathcal{B}} \longrightarrow \mathbb{R}$ *is an additive function such that, for every placement* μ*, we have*

$$H^{\mathrm{i}}(\mathcal{P}, \mathcal{Q}) = -\int_{\mathrm{Rct}_{\mu}(\mathcal{P},\mathcal{Q})} \frac{\mathbf{q}_{\mu}}{\theta_{\mu}} \cdot \mathbf{n}_{\mathcal{P}_{\mu}} \quad \text{for all } \ (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \ \text{ with } \ \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}} \ . \tag{11.1}$$

and

$$H^{\mathrm{e}}(\mathcal{P}) = \int_{\mathcal{P}_{\mu}} \frac{\rho_{\mu}(r \circ \mu^{\leftarrow})}{\theta_{\mu}} \qquad \text{for all } \ \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}} \ , \tag{11.2}$$

where $\mathbf{q}_{\mu}$ *is the heat flux in the placement* μ*,* r *is the specific external heat absorption density and* $\theta_{\mu} := \theta \circ \mu^{\leftarrow}$*. The pair* $(H^{\mathrm{i}}, H^{\mathrm{e}})$ *is called the* **entropy transfer system** *generated by the heat transfer system* $(Q^{\mathrm{i}}, Q^{\mathrm{e}})$ *and the temperature* θ*.* H^{i} *is called the* **internal entropy transfer***, and* H^{e} *is called the* **external entropy absorption**.

Proof: Let μ and μ' be placements and define a entropy transfer system $(H^{\mathrm{i}}, H^{\mathrm{e}})$ using the placement μ so that (11.1) and (11.2) hold. Put $\alpha := \mu \circ \mu'^{\leftarrow}$. By (7.4) and (11.1) we have

$$H^{\mathrm{i}}(\mathcal{P}, \mathcal{Q}) = -\int_{\mathrm{Rct}_{\mu'}(\mathcal{P},\mathcal{Q})} \frac{\rho_{\mu',\mu}(\nabla\alpha)^{-1}\mathbf{q}_{\mu} \circ \alpha}{\theta_{\mu} \circ \alpha} \cdot \mathbf{n}_{\mathcal{P}_{\mu'}} \quad \text{for all } \ (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \ \text{ with } \ \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}} \ . \tag{11.3}$$

From the discussion following (7.4) we know that the contactor $\mathbf{q}_{\mu'}$ of Q^{i} in the placement μ' is given by $\mathbf{q}_{\mu'} = \rho_{\mu',\mu}(\nabla\alpha)^{-1}\mathbf{q}_{\mu} \circ \alpha$ and, since $\mu = \alpha \circ \mu'$, we have $\theta_{\mu} \circ \alpha = \theta_{\mu'}$. Thus (11.3) becomes

$$H^{\mathrm{i}}(\mathcal{P}, \mathcal{Q}) = -\int_{\mathrm{Rct}_{\mu'}(\mathcal{P},\mathcal{Q})} \frac{\mathbf{q}_{\mu'}}{\theta_{\mu'}} \cdot \mathbf{n}_{\mathcal{P}_{\mu'}} \quad \text{for all } \ (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \ \text{ with } \ \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}} \ . \tag{11.4}$$

Equation (11.4) shows that H^{i} would be the same if it were defined in terms of the placement μ'. Since μ and μ' were arbitrary placements, the definition of H^{i} doesn't depend on the placement.

The proof that the definition of H^{e} doesn't depend on the placement is analogous to the one just given for H^{i} except the change of placement formula for densities (7.2) is used. ▮

Let a placement μ and a temperature θ be given. By definition, we have $\theta_\mu \circ \mu = \theta$. Taking the gradient of this equation at $X \in \mathcal{B}$, using the chain rule and (4.4), we obtain

$$\boldsymbol{\gamma}(X) := \nabla_X \theta = (\nabla_{\mu(X)}\theta_\mu)\nabla_X \mu = (\nabla_{\mu(X)}\theta_\mu)\mathbf{M}_\mu(X) \in \mathcal{T}^* \quad \text{for all} \;\; X \in \mathcal{B}\;. \tag{11.5}$$

Let δ denote the configuration associated with μ, as defined in (4.1). Then (11.5) and (10.5) can be used to obtain

$$\boldsymbol{\gamma}(X)\mathbf{h}_\delta(X) = (\nabla_{\mu(X)}\theta_\mu) \cdot \mathbf{q}_\mu(\mu(X)) \quad \text{for all} \;\; X \in \mathcal{B}\;. \tag{11.6}$$

Definition 16. *Let by a time-famiy $((\bar{Q}^{\mathrm{i}}_t\ , \bar{Q}^{\mathrm{e}}_t) \mid t \in I)$ of heat transfer systems and a time-family of temperatures $(\bar{\theta}_t \mid t \in I)$ be given, and let $((\bar{H}^{\mathrm{i}}_t\ , \bar{H}^{\mathrm{e}}_t) \mid t \in I)$ be the resulting entropy transfer system as described in Theorem 7. Let $(\bar{N}_t : \Omega_{\mathcal{B}} \longrightarrow \mathbb{R} \mid t \in I)$ be a differentiable time-family of additive mappings, called the* **internal entropy**. *We say that the family $((\bar{H}^{\mathrm{i}}_t\ , \bar{H}^{\mathrm{e}}_t, \bar{N}_t) \mid t \in I)$ is a* **dissipative entropical process** *if*

$$\bar{N}^\bullet_t(\mathcal{P}) \geq \mathrm{Res}_{\bar{H}^{\mathrm{i}}_t}(\mathcal{P}) + \bar{H}^{\mathrm{e}}_t(\mathcal{P}) \quad \text{for all} \;\; \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}} \text{ and } t \in I\;. \tag{11.7}$$

Note that the time-family of temperatures can be identified with a function $\bar{\theta} : \mathcal{B} \times I \longrightarrow \mathbb{P}^\times$. From now on we assume that, for each $t \in I$, $\bar{N}_t$ has densities. As in the previous section, we assume that a reference mass $m : \Omega_{\mathcal{B}} \longrightarrow \mathbb{P}^\times$, as described in Section 7, is given. Let $\bar{\eta} : \mathcal{B} \times I \longrightarrow \mathbb{R}$ be the mapping defined by the condition that $\bar{\eta}_t$ is the specific density of $\bar{N}_t$ for all $t \in I$. We call this mapping the **specific entropy** and we will assume that it is of class C^1.

Now let a motion $\bar{\mu}$ in a given frame-space $\mathcal{F}$, with translation space $\mathcal{V}$, be given. Of course, $\bar{\theta}$ is a material field. The spatial field $\bar{\theta}_{\mathrm{s}}$ associated with $\bar{\theta}$, according to (6.2), satisfies

$$(\bar{\theta}_{\mathrm{s}})_t := \bar{\theta}(t)_{\bar{\mu}(t)} \qquad \text{for all} \;\; t \in I\;. \tag{11.8}$$

If we let $\bar{\mathbf{q}}_t$ denote the heat flux in the placement $\bar{\mu}_t$ it follows from (11.1) and (11.2) that

$$\bar{H}^{\mathrm{i}}_t(\mathcal{P},\mathcal{Q}) = -\int_{\mathrm{Rct}_{\bar{\mu}(t)}(\mathcal{P},\mathcal{Q})} \frac{\bar{\mathbf{q}}_t}{(\bar{\theta}_{\mathrm{s}})_t} \cdot \mathbf{n}_{\mathcal{P}_{\bar{\mu}(t)}} \qquad \text{for all} \;\; (\mathcal{P},\mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \;\; \text{with} \;\; \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}}\;. \tag{11.9}$$

and

$$\bar{H}^{\mathrm{e}}_t(\mathcal{P}) = \int_{\mathcal{P}_{\bar{\mu}(t)}} \frac{\bar{\rho}_t(\bar{r}_{\mathrm{s}})_t}{(\bar{\theta}_{\mathrm{s}})_t} \qquad \text{for all} \;\; \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}}\;. \tag{11.10}$$

Using a line of reasoning analogous to the one that led from (10.6) to (10.8) in the previous section and using Theorem 5 and (11.7) and (11.8), we find that (11.7) is equivalent to

$$\bar{\rho}(\bar{\eta}^\bullet)_{\mathrm{s}} \geq \frac{\bar{\rho}\bar{r}_{\mathrm{s}}}{\bar{\theta}_{\mathrm{s}}} - \mathrm{div}\left(\frac{\bar{\mathbf{q}}}{\bar{\theta}_{\mathrm{s}}}\right)\;. \tag{11.11}$$

Hence, using Proposition 1 in Section 67 of [FDS], we obtain

$$\bar{\rho}(\bar{\eta}^\bullet)_{\mathrm{s}} \geq \frac{\bar{\rho}\bar{r}_{\mathrm{s}}}{\bar{\theta}_{\mathrm{s}}} - \frac{1}{\bar{\theta}_{\mathrm{s}}}\mathrm{div}\bar{\mathbf{q}} + \frac{1}{\bar{\theta}^2_{\mathrm{s}}}(\nabla\bar{\theta}_{\mathrm{s}}) \cdot \bar{\mathbf{q}}\;. \tag{11.12}$$

Definition 17. *A* **dynamical process** *is an octuple*

$$(\bar{\delta}, \bar{\theta}, \bar{\mathbf{F}}^{\mathrm{i}}, \bar{Q}^{\mathrm{i}}, \bar{E}, \bar{N}, \bar{\mathbf{F}}^{\mathrm{e}}, \bar{Q}^{\mathrm{e}}) \tag{11.13}$$

such that $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t, \bar{Q}^{\mathrm{i}}_t, \bar{Q}^{\mathrm{e}}_t, \bar{E}_t) \mid t \in I)$ is an energetical process as defined by Definition 15, $\bar{\theta}$ is a temperature process and $\bar{N}$ an internal entropy as used in Definition 16.

We assume that such a dynamical process is given and that all the density assumptions made before are satisfied, so that both (10.8) and (11.12) are valid.

By multiplying both sides of the inequality (11.12) by $\bar{\theta}_{\rm s}$ and using (10.8) to eliminate $\bar{\rho}\bar{r}_{\rm s} - {\rm div}\bar{\mathbf{q}}$ we obtain

$$\bar{\rho}(\bar{\theta}\bar{\eta}^\bullet - \bar{\epsilon}^\bullet)_{\rm s} + {\rm tr}(\bar{\mathbf{T}}\bar{\mathbf{D}}) - \frac{1}{\bar{\theta}_{\rm s}}\nabla\bar{\theta}_{\rm s}\cdot\bar{\mathbf{q}} \geq 0 \ . \tag{11.14}$$

Given $X \in \mathcal{B}$ and $t \in I$, let $\bar{\mathbf{h}}_t(X) \in \mathcal{T}_X$ denote the intrinsic heat flux at X in the configuration $\bar{\delta}_t$ and put $\bar{\gamma}_t(X) := \nabla_X\bar{\theta}_t \in \mathcal{T}_X^*$. It follows from (11.6) that

$$(\bar{\gamma}_t\bar{\mathbf{h}}_t)(X) = \bar{\gamma}_t(X)\bar{\mathbf{h}}_t(X) = ((\nabla\bar{\theta}_{\rm s})_t\cdot\bar{\mathbf{q}}_t)(\bar{\mu}_t(X)) \in \mathbb{R} \quad \text{for all} \ \ t \in I \ \ , \ X \in B \ . \tag{11.15}$$

If we replace the left side of (11.14) by its associated material field, using (9.12) and (11.15), we obtain

$$\bar{\rho}_{\rm m}(\bar{\theta}\bar{\eta}^\bullet - \bar{\epsilon}^\bullet) + \frac{1}{2}{\rm tr}(\bar{\mathbf{S}}\bar{\mathbf{G}}) - \frac{1}{\bar{\theta}}\bar{\gamma}\bar{\mathbf{h}} \geq 0 \ . \tag{11.16}$$

By (7.14), $\bar{\rho}_{\rm m}(X,t) = \rho_{\bar{\mu}_t}(\bar{\mu}_t(X)) = \rho_{{\rm imb}_{\rm t}}({\rm imb_t(X)})$ for all $(X,t) \in \mathcal{B}\times I$ and so $\bar{\rho}_{\rm m}$ only depends on the deformation process and not the motion. Thus, (11.16) does not involve any external frames of reference. Also, (11.16) does not depend on the particular choice of a reference mass. If one uses a different reference mass then the value of $\bar{\rho}_{\rm m}(X,t)$ would change by a strictly positive factor but both $\bar{\eta}^\bullet(X,t)$ and $\bar{\epsilon}^\bullet(X,t)$ would change by the reciprocal of this same factor, for all $(X,t) \in \mathcal{B}\times I$. Thus, the left side of (11.16) would remain the same.

12. Constitutive Laws and The Second Law of Thermodynamics

Definition 18. *A* **thermodeformation process** *is a pair*

$$(\bar{\delta}, \bar{\theta}) \tag{12.1}$$

in which $\bar{\delta} : \mathcal{B}\times I \longrightarrow {\rm Conf}\ \mathcal{B}$ *is a deformation process as defined in Definition 12 and* $\bar{\theta} : \mathcal{B}\times I \longrightarrow \mathbb{P}^\times$ *is a temperature process, which can be identified with a time-family of temperatures.*

A **response process** *is an quadruple*

$$(\bar{\mathbf{F}}^{\rm i}, \bar{Q}^{\rm i}, \bar{E}, \bar{N}) \tag{12.2}$$

where $\bar{\mathbf{F}}^{\rm i}$ *is an internal force system process, defined according to Definition 11,* $\bar{Q}^{\rm i}$ *is a internal heat transfer process, defined according to Definition 14,* $\bar{E}$ *is an internal energy as defined in Definition 15, and* $\bar{N}$ *is an internal entropy as defined in Definition 16.*

A **thermomechanical process** *is a hextuple*

$$(\bar{\delta}, \bar{\theta}, \bar{\mathbf{F}}^{\rm i}, \bar{Q}^{\rm i}, \bar{E}, \bar{N}) \ , \tag{12.3}$$

where $(\bar{\delta}, \bar{\theta})$ *is a thermodeformation process and* $(\bar{\mathbf{F}}^{\rm i}, \bar{Q}^{\rm i}, \bar{E}, \bar{N})$ *is a response process.*

Note that every thermomechanical process can be used to generate a dynamical process by using the balance of forces to *determine* the external force system process and the balance of energy to *determine* the external heat transfer process needed to produce the dynamical process.

Constitutive laws are used to describe the internal properties of a system and the internal interactions between its parts. In the framework presented here this means that given a set of constitutive laws each deformation process can be used to generate a response process and hence a thermomechanical process. All thermomechanical processes generated in this way are called **admissible** with respect to the given set of constitutive laws. The act of constructing admissible thermomechanical process from a set of constitutive laws will be carried out systematically by Brian Seguin in his doctoral thesis.

We are now in a position to state the final fundamental law of thermomechanics.

Second Law of Thermodynamics: Given a set of constitutive laws, every admissible thermomechanical process must satisfy the reduced dissipation inequality (11.15).

This law is a restriction on the set of constitutive laws, *not* on the class of thermodeformation processes a body can under go. There is an enormous amount of literature on this subject. See, for example, [CN],

[C] or [CO] and the references listed above. The restrictions found using this law are more easily expressed if one introduces the following concept.

Definition 19. *The* **specific free energy** *is a* C^1 *mapping* $\bar{\psi} : \mathcal{B} \times I \longrightarrow \mathbb{R}$ *defined by*

$$\bar{\psi} := \bar{\epsilon} - \bar{\theta}\bar{\eta} \; . \tag{12.4}$$

The **internal free energy process** $\bar{E}^{\mathrm{f}} : \Omega_{\mathcal{B}}^{\mathrm{int}} \times I \longrightarrow \mathbb{R}$ *associated with the specific free energy* $\bar{\psi}$ *is a differentiable time-family of additive mappings with densities defined by*

$$\bar{E}^{\mathrm{f}}_t(\mathcal{P}) := \int_{\mathcal{P}} \bar{\psi}_t \mathrm{d}m \quad \text{for all } \; t \in I \text{ and } \mathcal{P} \in \Omega_{\mathcal{B}}^{\mathrm{int}} \; . \tag{12.5}$$

Using the time derivative of (12.4) and (11.16) we obtain the **reduced dissipation inequality**

$$-\bar{\rho}_{\mathrm{m}}(\bar{\psi}^{\bullet} + \bar{\theta}\bar{\eta}^{\bullet}) + \frac{1}{2}\mathrm{tr}(\bar{\mathbf{S}}\bar{\mathbf{G}}) - \frac{1}{\bar{\theta}}\bar{\gamma}\bar{\mathbf{h}} \geq 0 \; . \tag{12.6}$$

Constitutive laws can change from point to point and are local[8] in the sense that at a material point X they should only involve arbitrary small neighborhoods of X in $\mathcal{B}$. We say that the body $\mathcal{B}$ consists of a **simple material** if the constitutive laws for every point $X \in \mathcal{B}$ involve only the body element $\mathcal{T}_X$. Most material properties of real materials are covered by the theory of simple materials.

13. External Influences

External influences specify how the environment influences the behavior of the body. The description of these external influences depend on the choice of an external frame of reference. Perhaps the most important of these external influences are boundary conditions. In most cases, an important external influence is **inertia**. The total external body force density can be written as a sum

$$\bar{\mathbf{b}} = \bar{\mathbf{b}}_{\mathrm{ni}} + \bar{\mathbf{b}}_{\mathrm{i}} \; , \tag{13.1}$$

where $\bar{\mathbf{b}}_{\mathrm{ni}}$ denotes the external body force density that comes from non-inertial forces, and $\bar{\mathbf{b}}_{\mathrm{i}}$ is the inertial body force density.

When an inertial frame of reference is used, then $\bar{\mathbf{b}}_{\mathrm{i}}$ is given by

$$\bar{\mathbf{b}}_{\mathrm{i}} = -\bar{\rho}\bar{\mathbf{a}} \; , \tag{13.2}$$

where $\bar{\rho}$ gives the inertial mass density at each point of the trajectory. However, if a non-inertial frame of reference is used, then the inertial body force density is given by the more complicated formula

$$\bar{\mathbf{b}}_{\mathrm{i}} = -\bar{\rho}(\ddot{\bar{\mathbf{u}}} + 2\bar{\mathbf{A}}\dot{\bar{\mathbf{u}}} + (\bar{\mathbf{A}}^{\bullet} - \bar{\mathbf{A}}^2)\bar{\mathbf{u}}) \tag{13.3}$$

where $\bar{\mathbf{u}}$ is a mapping whose value gives the position vector of a material point relative to a reference point which is at rest in some inertial frame. The mapping $\bar{\mathbf{A}}$, whose range consists of skew lineons, describes the motion of the non-inertial frame relative to the inertial frame. The second and third terms in the above formula are called the *Coriolis* force and *centrifugal* force, respectively. (See Part 2, Section 3, of [FC].)

Remark 14: Substituting (13.1) into the second term on the left of (9.10) one obtains

$$\int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot \bar{\mathbf{b}}_t = \int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot (\bar{\mathbf{b}}_{\mathrm{ni}})_t + \int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot (\bar{\mathbf{b}}_{\mathrm{i}})_t \; . \tag{13.4}$$

When one is using an inertial frame of reference (13.2) holds and the second term on the right is given by

$$\int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot (\bar{\mathbf{b}}_{\mathrm{i}})_t = -\left(\int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\rho}_t |\bar{\mathbf{v}}_t|^2\right)^{\bullet} \; . \tag{13.5}$$

[8] In [NLFT] this was called the *principle of local action.*

The term $\int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\rho}_t|\bar{\mathbf{v}}_t|^2$ is called the *kinetic energy.* Substituting (13.5) into (9.10) one obtains

$$\int_{\mathrm{Rby}\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t\cdot\bar{\mathbf{T}}_t\mathbf{n}_{\mathrm{Rby}\mathcal{P}_{\bar{\mu}_t}} + \int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t\cdot(\bar{\mathbf{b}}_{\mathrm{ni}})_t = \int_{\mathcal{P}_{\bar{\mu}_t}} \mathrm{tr}(\bar{\mathbf{T}}_t\bar{\mathbf{D}}_t) + \left(\int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\rho}_t|\bar{\mathbf{v}}_t|^2\right)^{\bullet} \quad \text{for all } \mathcal{P}\in\Omega_{\mathcal{B}} \text{ and } t\in I\,. \tag{13.6}$$

In the literature on continuum mechanics it is often implicitly assumed that the frame of reference being used is inertial so the formula (13.6) is valid. However, when the frame of reference is not inertial then (13.6) is not valid and the concept of kinetic energy is not useful. ▮

Constitutive laws can be specified using an external frame of reference. These laws would give the Cauchy stress $\bar{\mathbf{T}}$, spatial description of the specific free energy $\bar{\psi}_{\mathrm{s}}$, heat flux $\bar{\mathbf{q}}$ and spatial description of the specific entropy $\bar{\eta}_{\mathrm{s}}$ in terms of a thermodeformation process. Such constitutive laws would implicitly depend on the frame being used. Such dependence should be ruled out using the **Principle of Material Frame-Indifference**[9]. It states:

> Constitutive laws should not depend on whatever external frame of reference is used to describe them.

Traditionally one would specify a constitutive law in some frame of reference and then have to go though the effort of finding what restrictions are placed on this law by the principle of material frame-indifference. For some constitutive laws this can take a considerable amount of work. The way to eliminate this work is to formulate constitutive laws without using any external frames of reference. This can be done by specifying constitutive laws for the intrinsic stress, intrinsic heat flux and the specific free energy and specific entropy since these don't depend on the choice of a frame of reference. This superior method was used by W.N. in [N5] and [N6] and will be used systematically by Brian Seguin in his doctoral thesis.

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