4-23-2009

Filling in the Gaps: Creating an Online Tutor for Fractions

Nicole Hallinen
Carnegie Mellon University

Filling in the Gaps: Creating an Online Tutor for Fractions

Nicole Hallinen

Carnegie Mellon University

Abstract

Evidence has shown that students have significant difficulties in learning about the many representations of fractions. Because of the importance of fractional and proportional reasoning in more advanced mathematical study, such as algebra, an online tutor was developed to help children better understand fractions. Magnitude representations, specifically number line estimation practice questions, were highlighted in this tutor to emphasize the measurement property of fractions and the relationships between fractions and whole numbers. A study in which the type of fractions presented and practiced is manipulated between subjects is proposed to explore the relationship between fraction instruction and transfer to other fraction estimation contexts. It is hypothesized that presenting mixed numbers (fractions greater than one) will elicit better transfer to other types of fractions than presenting proper fractions (fractions less than one).

Introduction

In the scientific and educational community, it is widely agreed that learning about fractions and how to use them are difficult yet important conceptual and procedural skills for students to master. As quoted in a recent report of the National Mathematics Advisory Panel (NMAP), according to recent National Assessment of Educational Progress (NAEP) reports, "27% of eight-graders could not correctely shade 1/3 of a rectangle and 45% could not solve a word problem that involved dividing fractions" (U.S. Department of Education, 2004, qtd. in National Mathematics Advisory Panel, 2008, p. 3). In their 2008 final report, the NMAP issued several recommendations whereby American educational policy and mathematics teaching practices should be revised in an effort to improve mathematical literacy among American students. The Panel's report placed special emphasis on learning about fractions, suggesting new standardized teaching schedules specifically concerning fractions. The twelfth recommendation item states, "Difficulty with fractions … is a pervasive and major obstacle to further progress in mathematics, including algebra" (National Mathematics Advisory Panel, 2008, p. xix).

In the section of the report about fractions, the authors express the need for continued research and design of new instructional materials, particularly those that emphasize part-whole relationships to highlight the conceptual bases of fractional arithmetic procedures, which are often executed without understanding the underlying representations. These suggestions lead to a recommendation that curriculum designers plan to include both conceptual and procedural information about fractions with enough time to ensure that children learn the required basic material needed to advance to more complicated mathematical tasks. (National Mathematics Advisory Panel, 2008).

On a theoretical level, Kieren (1976) proposed five ways in which one could conceptualize fractions – part-whole, ratio, operator, quotient, measure – and it is suggested that this multifaceted nature makes it difficult both to teach and to learn fractions and proportional reasoning. The part-whole property of fraction representation indicates that fractions can be considered as a way of representing part of a whole set of objects or complete objects. Items to test this type of thinking typically include partitioning a shape into equal parts or determining how many objects would be in a whole set when given a particular part of a set. Thinking of fractions as ratios and rates implies "a comparison between two quantities; therefore it is considered a comparative index, rather than a number" (Charalambous & Pitta-Pantazi, 2007). Thus, tasks encouraging a ratio understanding of fractions require learners to determine relative amounts, like computing the relative amount of a set of food objects that each person would receive if the food were shared equally. The operator subconstruct of fractions refers to the multiplicative property of fractions. Drawing on arithmetic skill, students must conceptualize fractions as an indicator of multiplicative action to be applied to other numbers. Thinking of fractions as quotient forces one to consider fractions as "the result of a division situation" (Charalambous & Pitta-Pantazi, 2007). Finally, the measure subconstruct is comprised of two underlying concepts: fractions are considered quantitative, as numbers that indicate magnitudes, and fractions are considered a measurement of a particular interval (Charalambous & Pitta-Pantazi, 2007). This subconstruct has been widely tested through the use of number line estimation and ruler measurement tasks. Many researchers claim that considering fractions as unique numbers as opposed to two a combination of whole numbers is difficult for children because the non-whole number property of fractional magnitudes violates principles of the counting sequence that children initially form (Charalambous & Pitta-Pantazi, 2007).

Using a 50-item test, Charalambous & Pitta-Pantazi (2007) conducted a factor analysis to understand the relations among these subconstructs and three operations: equivalence, multiplicative, and additive. They found that the part-whole subconstruct mediated the other four subconstructs but that further research is needed to fully understand these relationships and the various subtleties of each of the five subconstructs.

Despite the identification of this fundamental problem in children's mathematical understanding, there is little agreement on an appropriate instructional response. Thus, several fraction-teaching curricula exist. Keijer and Terwel (2003) explicitly investigated the advantages and disadvantages of two fractions curricula: one based on number line representations and a "fair-sharing" model. The "fair-sharing" model, similar to that developed by Streefland (1990), involves framing fractional understanding in terms of fairly sharing a given whole number quantity among a group of people. This representation, a ratio-centered model, is typically shown in textbooks and other teaching materials as dividing a pizza or pie into an appropriate number of equal pieces to share among a group of people. Keijer and Terwel's number line curriculum consisted of using three cover stories to explain the role of fractions in describing numerical magnitudes. First, a measurement unit known as the "Amsterdam Foot" was created such that children had to use fractional parts of this unit to measure other items. Second, a "Treasure-Hunting" game required students to identify the locations of fractions along a horizontal plane to determine the location of a hidden treasure. Third, a vertical number line was employed in an elevator game in which participants had to note equivalent fractions as the same position in an elevator shaft. Through a year-long longitudinal experiment, the researchers concluded that the experimental number line group significantly outperformed the fair-sharing and part-whole control group on periodic standardized tests and that the students in the experimental group performed better in qualitative measures of conceptual understanding like drawings and explanations of the meaning of fractions. (Keijer & Terwel, 2003)

One common way to highlight the Kieren's (1976) measurement subconstruct of fractions is by using number line representations to demonstrate the relations between fractions and whole numbers as existing along the same continuum at various magnitudes. Number line estimation tasks have also been shown to provide effective means for instructing students about decimal representations of fractions. As the National Mathematics Advisory Panel Report states, "One key mechanism linking conceptual and procedural knowledge is the ability to represent fractions on a number line" (National Mathematics Advisory Panel, 2008, p. xix). Rittle-Johnson, Siegler, & Alibali (2001) created a game entitled "Catch the Monster" in which participants selected an appropriate fractional location on a number line to find a hiding monster. Using this game in practice sessions with elementary school students, these authors demonstrated a link between number line practice on decimal fractions less than 1 and performance on an assessment of students' conceptual understanding of fractions that required students to explain why a given decimal would be found at a particular location on a number line. Students also showed some transfer to a later decimal fractions estimation task that involved a 0 to 10 number line scale. An iterative model in which procedural and conceptual knowledge continually build upon one another was proposed from these data. (Rittle-Johnson et al., 2001)

A major obstacle involved in using fair-sharing or other part-whole models is the difficulty of translating this type of representation to improper fractions or mixed numbers – fractional numbers greater than one. As the National Mathematics Advisory Panel Report stated, "Conceptual and procedural knowledge about fractions with magnitudes less than 1 do not necessarily transfer to fractions with magnitudes greater than 1" (2008, p. 28). Because little work has been done to understand the possible benefits of teaching children about larger, improper fractions and mixed numbers directly, two sets of instructional materials were designed to specifically address this question of an ideal order in which to present these two types of fractions.

Design of Computer-Based Instructional Materials

In the present investigation, computer-based instructional materials were designed for use by elementary school students to practice magnitude representations of fractions. The instructional materials include a short informational section about fractions followed by a fraction magnitude practice section that requires learners to locate fractional magnitudes along a number line.

In this instructional practice section, two forms of feedback are provided to encourage student reflection (see Figure 1). First, the tutor revealed a percent accuracy score after each practice question that was calculated using the formula (1 – percent error) * 100. Therefore, the maximum score on an individual question was 100%. Additionally, the hatch mark indicating the correct location for each fraction is displayed in a dark blue color on the number line while a line denoting the participant's response is displayed on a color gradient that corresponds to the correctness of his or her click. That is, a very close attempt is much more similar in color to the dark blue of the correct answer and a less close attempt is represented by a lighter blue line. Students are encouraged to compare their selections with the correct answer through a textual reminder that appears on the screen after each attempt. Students are not permitted to retry questions they have already answered.

Two versions of this instructional software have been created. In one condition, mixed numbers (fractions greater than one) are presented in practice questions that require students to estimate the magnitudes of these numbers on a number line ranging from 0 to 5. Conversely, the other version requires estimation of proper fractions (those less than one) on a number line scaled from 0 to 1. The other instructional materials and feedback structures are identical across these two conditions.

Identical 16-item pre- and post-tests were also constructed to test students' knowledge of fractional magnitudes, again using a number line estimation task. Four blocks of four fractions each comprise these 16 fractions: (i) fractions less than 1 on a number line ranging from 0 to 1, (ii) fractions less than 1 on a number line ranging from 0 to 5, (iii) mixed number fractions between 1 and 5 on a number line ranging from 0 to 5, (iv) fractions ranging from 0 to 10 on a number line ranging from 0 to 10. Participants' performance on fractions presented in block (i) will serve as a direct post-test on the practiced material for learners in the proper-fraction practice condition and indicate possible transfer for individuals in the mixed-number practice condition. Conversely, block (iii) includes mixed numbers that mixed-number practice students will have practiced while offering near transfer questions for the proper-fraction condition. Block (ii)'s composition enables the experimenter to determine possible differences in transfer of magnitude representations to different scales, as in the case of the proper-fraction condition using a new scale, and transfer to a new range of numbers as the mixed-number condition will be forced to estimate new types of numbers in the same broader number line context. Finally, all participants will attempt far transfer questions in block (iv) as both the number line scale and magnitudes for estimation will be unfamiliar for both conditions. The four fractions within each block are presented in a random order but all four fractions within a block are presented in succession. Moreover, each block is presented in a random order during the pre- and post-tests.

In both the pre- and post-test materials and the instructional practice module, the selection of example fractions was constrained as follows: (a) all numerators and denominators ranged from one to nine, (b) only reduced fractions were used (e.g. 2/6 would not be acceptable), (c) 1/2 was not used. (See Appendix A.) The range of numbers included as numerators and denominators was limited to ensure that late elementary-school aged children would be familiar with all numbers presented and to avoid the inclusion of very small fractions (e.g. 1/73) among which it may be difficult for learners to distinguish unique magnitudes. Reduced fractions were used to prevent the duplication of a particular magnitude in multiple estimation tasks (e.g., 2/3, 4/6, 6/9) as this exercise did not specifically address fraction equivalence. One-half was not used because of its overuse in daily conversation and its familiarity to students, which may lead to artificial adeptness with this magnitude in particular. On all number lines, whole number values were not delineated with any special markings; only the two endpoints were labeled. The size of the number line image on the computer screen was consistent across conditions such that a number line ranging from 0 to 1 was the same length as one ranging from 0 to 5 or 0 to 10.

These computer-based materials were created using Adobe Flash 8 on a Macintosh computer.

Proposed Experiment

Experimental Design

In a proposed experiment, participants will utilize these two learning environments in a between-subjects design to practice identifying a series of fractions on a number line. That is, one group will practice estimation tasks involving proper fractions (fractions less than one) and the other group will estimate mixed numbers (fractions greater than one). All participants will complete the same pre- and post-test measures.

Procedure

Each participant will begin by completing the pre-test questions as presented in a unique randomized order by the computer program (i.e., each test will be randomized independently). Next, he or she will continue by reading the instructional section and completing the practice estimation questions that correspond to the condition to which he or she will be randomly assigned. The participant will then complete the post-test questions. The entire session will take no longer than one hour.

Hypotheses

This experimental design will allow for an investigation into several types of transfer questions that involve new estimation tasks (see Table 1). First, by requiring participants to estimate numbers with which they have no estimation practice, these blocks of questions will allow researchers to determine the possibility of scaling understanding of smaller fractions to larger numbers and vice versa. Understanding that fractions have magnitudes that can be represented in the context of other numbers emphasizes the measurement property of fractions described by Kieren (1976). Additionally, participants will be asked to use unfamiliar number line scales, either smaller (in the case of the mixed number participants using the 0-1 scale) or larger (in the case of the proper fractions group using the 0-5 scale). The ability to transfer among different number line ranges reinforces the necessity of understanding the ratio property of fractions, and indeed all numbers. All participants will encounter both types of transfer when estimating a range of numbers on a 0 to 10 number line in the far transfer situation.

This experiment will be valuable in determining how transfer to different scales and to different ranges of magnitudes occurs. I hypothesize that the mixed number condition would lead to more effective transfer of fractional understanding to smaller fractions as compared to the proper fraction condition’s success in producing transfer to larger fractions. In particular, the mixed-number condition may outperform the proper fraction condition on the far transfer tasks (fractions between 0 and 10, number line 0 – 10) as generalizing between two ranges of mixed numbers may be more easily accomplished than generalizing proper fraction knowledge to mixed numbers.

Discussion

Conducting a study to determine if and how fraction estimation abilities can be transferred will be critical in addressing the question of developing an appropriate way to instruct children about mixed numbers and proper fractions.

References

Charalambous, C. Y. & Pitta-Pantazi, D. (2007). Drawing on a Theoretical Model to Study Students' Understandings of Fractions. *Educational Studies in Mathematics, 64*, 293-316.

Keijer, R. & Terwel, J. (2003). Learning for Mathematical Insight: A Longitudinal Comparative Study on Modeling. *Learning and Instruction, 13*, 285-304.

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and Measurement,* Columbus: Ohio State University, EEIC, SMEAC. pp. 101 – 144.

National Mathematics Advisory Panel. (2008). *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*, U.S. Department of Education: Washington, DC.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing Conceptual Understanding and Procedural Skill in Mathematics: An Iterative Process. *Educational Psychology, 93(2),* 346 – 362.

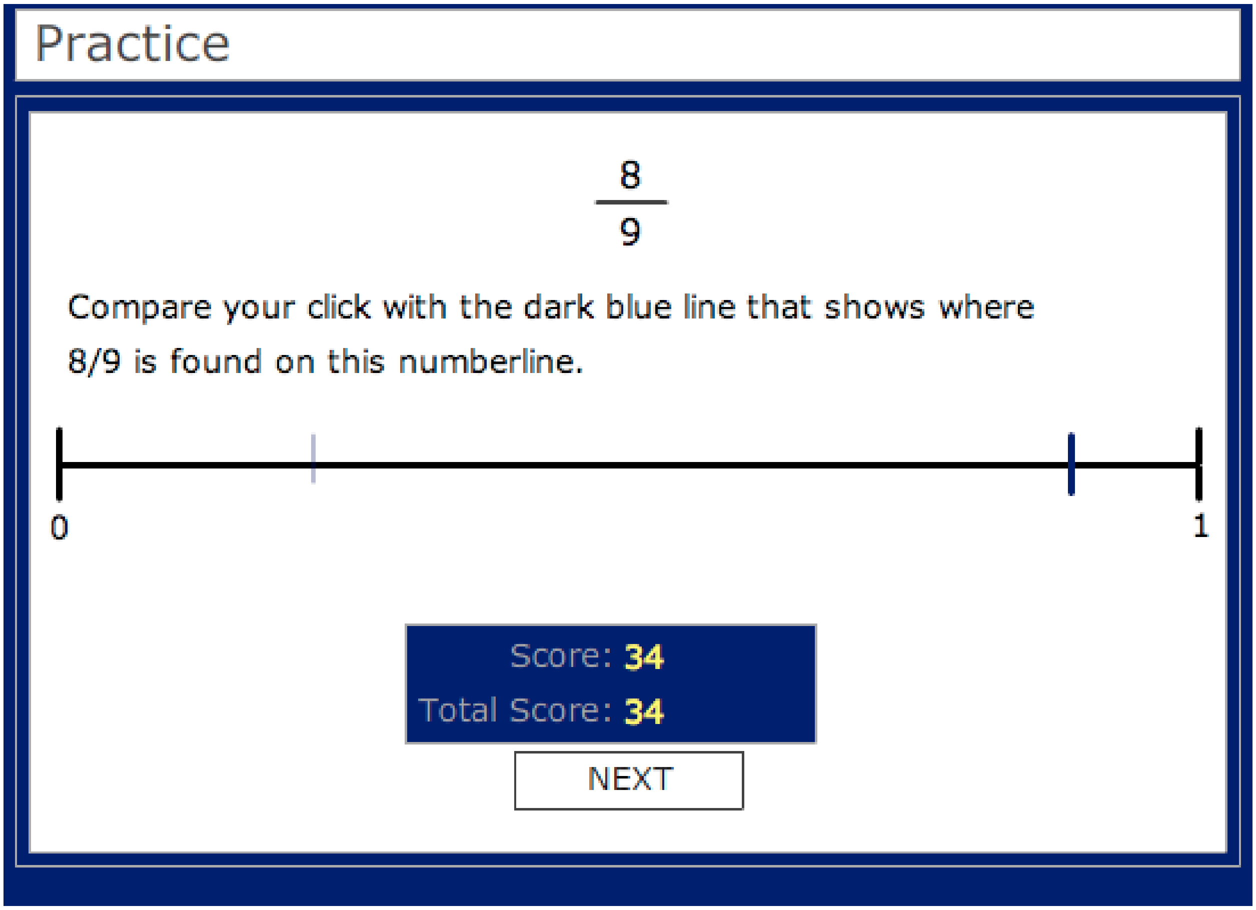
Streefland, L. (1990). Realistic Mathematics Education (RME). What does it mean?. In: K.P.E. Gravemeijer, M. van den Heuvel-Panhuizen & L. Streefland (Eds.), *Contexts, Free Productions, Tests and Geometry in Realistic Mathematics Education*, Utrecht: OW&OC, pp. 1-10.

Table 1.

Number line Fractions	0 – 1 Less than 1	0 – 5 Less than 1	0 – 5 1 – 5	0 – 10 0 – 10
Group A Small Fractions	Same as practice	New line Same numbers	New line New numbers	Far Transfer
Group B Mixed Numbers	New line New numbers	Same line New numbers	Same as practice	Far Transfer

Experimental Design: Between subjects practice session (Groups A and B) with identical pre- and post-tests comprised of four types of fraction estimation problems.

Figure 1.



Practice module interface depicting two forms of feedback: score display and colored answer comparison.

Appendix A: List of fractions used in pre- and post-test questions.

0 – 1 Number line, Fractions less than 1

1/5
2/3
1/8
3/4

0 – 5 Number line, Fractions less than 1

4/5
5/8
2/9
3/7

0 – 5 Number line, Fractions between 1 and 5

1 1/3
2 5/6
3 3/5
4 6/7

0 – 10 Number line, Fractions between 0 and 10

3/8
2 5/7
5 1/8
8 7/9