Epistemic Conditionals, Snakes, and Stars
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1 Background

Consider a rational agent $X$ at certain point of time $t$. $X$’s epistemic state can be represented in different ways. According to the Bayesian tradition, $X$’s state of belief should be represented by a probability measure defined over some language $L_0$, containing neither modal nor epistemic operators.

A cruder way of modelling $X$’s state of belief can be adopted, namely the one by which $X$’s epistemic state is represented by a set of sentences of $L_0$.\(^1\) The intended interpretation of such a belief set is that it consists of all the sentences which $X$ accepts at $t$.

We will adopt here the simpler model and will suppose also that the ideal of rationality embodied by $X$ requires the following: (1) $X$ is logically omniscient, but not necessarily omniscient, (2) $X$ is autoepistemically omniscient, i.e. $X$ knows what he or she knows and $X$ also knows what he or she does not know, (3) at any point of time potential epistemic changes of $X$’s view are represented by epistemic commitments, which are to be understood as functions that express how the current corpus should change in the presence of suppositional inputs of certain sort, and (4) $X$ is autoepistemically omniscient with respect to the potential outcomes of his policies for changing his or her view.\(^2\)

More precisely: (1) is satisfied by requiring that $X$’s epistemic state be represented by a theory $K$,\(^3\) which needs not be complete, (2) means that for every sentence $A \in L_0$, $X$ knows whether $A$ is accepted or not in $K$, and (3) suggests the existence of a function that yields a new epistemic state for any given state $K$ and a given suppositional input $A$. The potential input can contradict the sentences of the current belief set $K$. In this case, the function picks up a unique new epistemic state called the revision of $K$ by $A$ ($K^A$). Extensive research has been done on the characterization of $\ast$. Alchourrón et al. \[1\] have offered an

\(^1\)This type of model is formally presented by Peter Gärdenfors in \[14\], chapter 1.

\(^2\)A theory of rationality that satisfies at least all these constraints was defended by Isaac Levi in several writings (see the articles “Serious Possibility” and “Subjunctives, Dispositions and Chances in \[28\]\)

\(^3\)A theory is a set $K$ of sentences of $L_0$ such that every consequence of $K$ belongs to $K$.\]
analysis of belief revision (usually called AGM) that is considered standard in
the literature.\footnote{This does not mean that the discussion concerning the foundations of belief revision
text is closed. The validity of some postulates like the so-called recovery postulate have
been copiously discussed (see [30] and [31])}

Finally, the last rationality requirement can be interpreted as follows: sup-
pose that $X$ is committed to change from $K$ to $K^*B$ if he or she is presented
with the suppositional input $B$. Then $X$ should know for any sentence $A$, whether $A$
$\in K^*B$ or $A \not\in K^*B$.

Of course the last type of autoepistemic knowledge can be expressed using
conditional sentences (or their negations). For example the fact that we are
committed to accept $B$ in $K^*A$ is usually made public linguistically via locutions
of the form “If $A$, then $B$”. By the same token, the fact that $B$ does not belong
to $K^*A$ can be expressed by “It is not the case that if $A$, then $B$.”

These conditionals are not part of the stock of $X$’s “first order” beliefs, but
they are part of $X$’s metabeliefs about $X$’s own beliefs, and the ways that they
may change. We can call these conditionals \textit{epistemic conditionals}. This essay
is centrally devoted to studying the logical structure of such conditionals, and
their relevance for the formalization of nonmonotonic reasoning.

\section{1.1 Ramsey test}

We have already implicitly suggested criteria of acceptability for epistemic condi-
tionals. In fact, an epistemic conditional ‘If $A$ then $B.$’ will be accepted with
respect to the belief set $K$ iff $B$ belongs to $K^*A$. Similarly ‘It is not the case
that if $A$, then $B$’ will be accepted with respect to a belief set $K$ iff $B$ does not
belong to $K^*A$. If we use the symbol $\triangleright$ to represent the relation ‘... is acceptable
with respect to ...’, and we abbreviate ‘If $A$, then $B$’ by $A \triangleright B$, we can express
these criteria in a more compact way as follows:

\begin{align*}
\text{(RT) } & A \triangleright B \triangleright K \text{ iff } B \in K^*A \\
\text{(NRT) } & \lnot (A \triangleright B) \triangleright K \text{ iff } B \not\in K^*A.
\end{align*}

(RT) is known in the literature as the Ramsey test. It is so named because
an ancestor of this test was first suggested by Ramsey in a footnote to his article
“General Propositions and Causality” (see [37]). Of course, more has to be said
in order to specify fully the structure of (RT) and (NRT). Specifically, we need
a satisfactory characterization of $\triangleright$.

Gärdenfors [12] proposed a precise characterization. On his view sentences
that contain “$\triangleright$” can be elements of belief sets. In other words, belief sets
contain both modal and nonmodal sentences. Therefore the acceptance of $A \triangleright B$
with respect to a belief set $K$ is represented by the fact that $A \triangleright B$ belongs to
$K$. This implies that Gärdenfors’ formulation of the Ramsey test ((GRT) from
now on) can be obtained by substituting $\in$ for $\triangleright$ in (RT). Of course this move
conflicts with the metalinguistic nature of the epistemic conditionals suggested above. The identification of $\triangleright$ and $\in$ has some other serious consequences as well. In fact, Gärdenfors [14] proved that the change operator that appears in (GRT) cannot be AGM on pain of triviality. Moreover, as has been noted independently by many researchers (see [29] and [15]) the identification of $\triangleright$ and $\in$ leads to the abandonment of (NRT).

The intuitions that motivate our theory of epistemic conditionals are not coherent with Gärdenfors’ solution. In fact, the epistemic conditionals are merely the linguistic expression of our epistemic commitments which, in turn, are expressed by functions like AGM. Therefore we would like to use a test that is at least compatible with AGM. Moreover, we would also like to preserve (NRT) which is the general expression of the fourth rationality constraint presented above.

Levi [29] suggested a different analysis of $\triangleright$ that is in principle compatible with AGM and (NRT). He proposed using a support function $s$, which, when applied to a belief set $K$ expressed in a non-modal language $L_0$, yields the modal sentences supported by the set (indicated as $s(K)$). The support function obeys the following constraints.

(1) If $A \in L_0$, and $A \in s(K)$, then $A \in Cn(K)$, where $Cn$ is assumed to be a consequence operation defined over $L_0$.

(2) $K \subseteq s(K)$, whenever $K$ is consistent.

(3) $s(K)$ is closed under logical consequence.\(^5\)

(SRT) If $A, B \in L_0$, then $A \triangleright B \in s(K)$ iff $B \in K^*A$, whenever $K$ is consistent.

(SNRT) If $A, B \in L_0$, then $\neg (A \triangleright B) \in s(K)$ iff $B \not\in K^*A$, whenever $K$ is consistent.

Conditions (1) and (2) identify support and acceptability conditions for non-modalized (or “indicative”) sentences. Clause (3) establishes that “support sets” are closed under classical deduction. Finally (SRT) and (SNRT) specify the stratified versions of (RT) and (NRT).

Many alternatives to (GRT) have been explored in the literature, in addition to the one offered by Levi. None of them is compatible with AGM, and more importantly none of them represents in a better way the intuitive idea behind the use of epistemic conditionals. Nevertheless, no formal investigation of the structure of the conditionals delivered by (SRT) and (SNRT) has previously been offered. In particular no detailed analysis of the conditional systems generated by AGM has been provided in the literature. One of the main aims of this article is to provide such an analysis, and to show how it can be naturally applied to model non-monotonic notions of consequence.

\(^5\)See section 3 for a precise definition of the notion of consequence presupposed by this condition.
2 Belief revision

In this section we will formally characterize the notion of change known as AGM in the literature. Motivation and intuitive background for AGM can be found elsewhere (see for example chapters 3 and 4 of [14]).

Let $L_0$ be a language containing a complete set of Boolean connectives, including \textit{falsum} and \textit{verum} constants $\bot$ and $T$. The set of wff of $L_0$ are defined in the usual manner. In addition we will assume a compact\textsuperscript{6} and consistent\textsuperscript{7} notion of consequence (Cn) defined over $L_0$, satisfying: $A \rightarrow B$ belongs to Cn(K) iff B $\in$ Cn(K $\cup \{A\}$). We suppose also that Cn(\emptyset) contains all substitution instances of classical tautologies expressible in $L_0$, and that it satisfies the Tarskian postulates of Inclusion, Iteration and Monotonicity. Belief states of rational agents will be represented by \textit{belief sets} satisfying the following rational constraints:

\textbf{Definition 2.1} A belief set is a subset K of $L_0$ which satisfies the following conditions:

For every $A, B \in L_0$, (1) K is non-empty. (2) if $A \in K$ and $B \in K$, then $(A \land B) \in K$, and (3) if $A \in K$, and $A \rightarrow B \in \text{Cn}(\emptyset)$, then $B \in K$.

Let $K + A$ denote an \textit{indication expansion} of the belief set K $\subseteq L_0$ with the sentence $A \in L_0$. Expansions are defined as follows: $K + A = \text{Cn}(K \cup \{A\})$. Let T be the set of all belief sets that can be expressed in $L_0$. Belief revisions are modelled by a function from $T \times L_0$ to T, which for every belief set K and every sentence A yields a unique belief set $K^*A$ which represents the minimal change of K needed to include A consistently in K. Such functions are characterized in [14] via the following set of rationality postulates:

(K\textsuperscript{*1}) For any sentence A and belief set K, $K^*A$ is a belief set.

(K\textsuperscript{*2}) $A \in K^*A$.

(K\textsuperscript{*3}) $K^*A \subseteq K + A$.

(K\textsuperscript{*4}) If $A \notin K$, then $K + A \subseteq K^*A$.

(K\textsuperscript{*5}) If A is consistent, then $K^*A$ is consistent.

(K\textsuperscript{*6}) If $A \leftrightarrow B \in \text{Cn}(\emptyset)$, then $K^*A = K^*B$.

(K\textsuperscript{*7}) $K^*(A \land B) \subseteq (K^*A) + B$.

(K\textsuperscript{*8}) If $\neg B \notin K^*A$, then $(K^*A) + B \subseteq K^*(A \land B)$.

Postulates (K\textsuperscript{*1})-(K\textsuperscript{*4}), (K\textsuperscript{5}), and (K\textsuperscript{*6}) are usually called the \textit{basic} postulates of AGM. The following theorems will be useful later:

\textsuperscript{6}For all $X \subseteq L_0$, Cn($X$) = $\cup\{C(X')$: $X'$ is a finite subset of X\}

\textsuperscript{7}$\bot \notin \text{Cn}(\emptyset)$
(CM*) If $B \in K^*A$ and $C \in K^*A$, then $C \in K^*(A \land B)$.

(wp) If $A \in K$, and $K$ is consistent, then $K \subseteq K^*A$.

(K* 5) If $A$ is consistent, and $K$ is consistent, then $K^*A$ is consistent.

(K* 7') $K^*A \cap K^*B \subseteq K(A \lor B)$

In the following we will denote by AGM' the notion of change characterized by AGM minus (K* 5). In addition we will denote by AGM-wp any AGM' notion of revision where (K* 4) is replaced by (wp) and (K* 8) is substituted for a version of it that only holds for maximal and complete theories.

Another postulate that will be useful later is

(cso) If $A \in K^*B$, and $B \in K^*A$, then $K^*A = K^*B$.

We will call $B$ any AGM-wp notion of revision where (K* 6) is replaced by (cso).

3 The epistemic model

First we need some logical preliminaries.

Let $L_>$ be the smallest language such that: (1) $L_0 \subseteq L_>$, (2) If $A, B \in L_>$, then $A > B \in L_>$, and (3) $L_>$ is closed under the Boolean connectives.

Let $FL_>$ be the smallest language such that: (1) $L_0 \subseteq FL_>$ (2) if $A, B \in L_0$, then $A > B \in FL_>$ and (3) $FL_>$ is closed under the Boolean connectives. Let an f-instance (flat-instance) of a conditional formula of $FL_>$ be a substitution instance of the formula where formulas of $L_0$ are substituted for the variables-scheme that occur in the formula. In addition, we will call every conditional formula that belongs to $FL_>$, ‘flat’.

We will also need, in addition to the classical notion of consequence, a compact and consistent notion $C$ defined over $FL_>$ satisfying for all $A, B \in FL_>$ and $K \subseteq FL_>$; $A \rightarrow B$ belongs to $C(K)$ iff $B \in C(K \cup \{A\})$. $C$ also satisfies Superclassicality. $C(\emptyset)$ contains all classical tautologies and their substitution instances in the language $FL_>$, as well as Inclusion, Iteration and Monotonicity.$^8$

**Definition 3.0.1** A conditional support set is a subset $K$ of $FL_>$ which satisfies the following conditions: for every $A, B \in FL_>$, (1) $K$ is non-empty,

$^8$We will use also the Gentzen-like notation $A \vdash B$ to indicate $A \in C(A)$. In particular $\vdash A$ is an abbreviation of $A \in C(\emptyset)$. 


(2) if $A \in K$ and $B \in K$, then $(A \land B) \in K$, and (3) if $A \in K$, and $\vdash A \rightarrow B$, then $B \in K$.

Let $FT_\rightarrow$ be the set of all conditional support sets expressible in $FL_\rightarrow$.

Now we can introduce the Epistemic Models via the following definitions.

An Epistemic Model (EM) is a triple $\langle K, *, s \rangle$ where $K$ is a set of belief sets closed under indicative expansions and revisions, * is a belief revision function, $*: K \times L_0 \rightarrow K$, and $s$ is a “support function” $s: K \rightarrow FT_\rightarrow$, obeying (SRT), (SNRT), and the following conditions:

1. If $A \in L_0$, and $A \in s(K)$, then $A \in Cn(K)$.
2. $K \subseteq s(K)$, whenever $K$ is consistent.
3. $s(K) = C(s(K))$

The model can be extended with the following epistemic counterparts of the classical notions of satisfaction, validity and entailment.

**Definition 3.1** An $FL_\rightarrow$ sentence $A$ is satisfiable in an EM $\langle K, *, s \rangle$ if there is a consistent $K \in K$, such that $A \in s(K)$.

**Definition 3.2** A set of $FL_\rightarrow$-sentences $G$ is satisfiable in an EM $\langle K, *, s \rangle$ if there is a consistent $K \in K$, such that for every $A \in G, A \in s(K)$.

**Definition 3.3** An $FL_\rightarrow$ sentence $A$ is positively-valid (PV) in a $\langle K, *, s \rangle$ if $A \in s(K)$ for every consistent $K \in K$.

**Definition 3.4** An $FL_\rightarrow$ sentence $A$ is e-valid if it is PV in every model.

**Definition 3.5** For every expression $A_0, A_1, ..., A_n, A$ of $FL_\rightarrow$, $A_0, A_1, ..., A_n \vdash_+ A$ iff for every $K \in K$, such that $A_0 \in s(K), A_1 \in s(K), ..., A_n \in s(K)$; $A \in s(K)$.

Consider the conditional system $CM$. $CM$ is the smallest set of formulas in the language $L_\rightarrow$ containing the following axiom schemata and closed under the following rules of inference.

**T** All classical tautologies and their substitution instances in $L_\rightarrow$.

**I** $A > T$

**CC** $((A > B) \land (A > C)) \rightarrow (A > (B \land C))$

**RCM** If $\vdash B \rightarrow C$ then $\vdash (A > B) \rightarrow (A > C)$

**M** Modus ponens.
Now, consider the following flat-version of the rule of inference (RCM):

**RCMf** If \( \vdash B \rightarrow A \) and \( A, B, C \in L_0 \) then \( \vdash (A > B) \rightarrow (A > C) \).

Flat(\( \mathcal{CM} \)) can be now defined as the smallest set of formulas in the language \( FL_> \) which is closed under (RCMf) and (M), and which contains all \( f \)-instances of the axioms (I), (CC), and all classical tautologies and their substitution instances in the language \( FL_> \).

A completeness result showing the coincidence of the theses of the system Flat(\( \mathcal{CM} \)) and the flat conditionals positively validated by the EMs is proved immediately.

**Theorem 3.1** A conditional flat formula \( A \) is \( e \)-valid iff \( A \) is a theorem in Flat(\( \mathcal{CM} \)).

Proof: See Appendix A.

Let us now consider the system \( \mathcal{CM}^+ \), characterized as the smallest set of formulas in the language \( FL_> \) that contains every \( f \)-instance of (I), every \( f \)-instance of (CC), all classical tautologies and their substitution instances in the language \( FL_> \), and every instance of each schema

\( ((A > C) \leftrightarrow (B > C)) \), whenever \( \vdash A \leftrightarrow B \) and \( A, B, C \in L_0 \) (RCEA)

and that is closed under (M) and (RCMf).\(^9\)

**Observation 3.1** A formula is derivable in \( \mathcal{CM}^+ \) iff it is PV in the class of all EMs satisfying (K*6).

Proof: See Appendix B.

The following observations show which is the flat conditional counterpart of the AGM postulates (K*2) and (K*8).

**Observation 3.2** All \( f \)-instances of (ID): \( A > A \), are PV in an EM \( \mathcal{M} \) iff \( \mathcal{M} \) satisfies (K*2)

**Observation 3.3** Assume that (ID) is PV in a model \( \mathcal{M} \). Then all \( f \)-instances of (CV): \( ((A > C) \land \neg (A > \neg B)) \rightarrow ((A \land B) > C) \) are PV in an EM \( \mathcal{M} \) iff \( \mathcal{M} \) satisfies (K*8).

Now we will establish the flat conditional counterpart of the AGM postulates (K*7) and (CM*).\(^{10}\)

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\(^9\)We will presuppose \( (\)From now on that all the extensions of \( \mathcal{CM}^+ \) contain all classical tautologies and their substitution instances in \( FL_> \) (axiom T of \( \mathcal{CM} \)).

\(^{10}\)(K*7) and (CM*) are derived postulates of AGM. See section 2.
Observation 3.4 All f-instances of \((CC')\): \(((A > C) \land (B > C)) \rightarrow ((A \lor B) > C)\) are PV in an EM \(\mathcal{M}\) iff \(\mathcal{M}\) satisfies the basic postulates plus \((K^*7')\).

Proof: See Appendix B.

Observation 3.5 All f-instances of \((CM)\): \((A > B) \land (A > C) \rightarrow ((A \land B > C)\), are PV in an EM \(\mathcal{M}\) iff \(\mathcal{M}\) satisfies \((CM^*)\).

Theorem 3.1 together with Observations 3.1-4 are enough to characterize the “flat-counterpart” of the system \(\mathcal{N}\) of defeasible conditionals proposed by J. Delgrande in [8]. By the “flat-counterpart of \(\mathcal{N}\)” we understand the least set of conditional theses of \(FL_\geq\) closed under \(\mathcal{M}\), (RCEA) and (RCMf) and which contains every \(f\)-instance of the theses \((ID)\), \((CC')\), \((CV)\), and \((CC)\). The importance of the system \(\text{Flat}(\mathcal{N})\) will be evident in section 6 where the EMs are applied to model the standard systems of nonmonotonic preferential logic.

The flat counterpart of the minimal system of conditional logic characterized by David Lewis in [32], the system \(\mathcal{V}\), can be epistemically generated by Theorem 3.1 together with Observations 3.1-5.

The following observation determines the conditional thesis that mirrors the postulates \((K^*3)\) and \((K5)\):

Observation 3.6 All f-instances of \((MP)\): \((A > B) \rightarrow (A \rightarrow B)\) are PV in an EM \(\mathcal{M}\) iff \(\mathcal{M}\) satisfies \((K^*3)\).

Proof: See Appendix B.

Observations 3.1-6 together with Theorem 3.1 enable us to reconstruct epistemically the flat counterpart of another system of conditional logic previously proposed by David Lewis in [32]: the system \(\mathcal{VW}\). Finally by adding \((K5)\) we obtain the stronger system of epistemic flat’ conditionals \(\mathcal{E}\mathcal{F}\).\(^{11}\) The system \(\mathcal{E}\mathcal{F}\) is the smallest set that contains every \(f\)-instance of the axioms \((ID)\), \((CC),(CC')\), \((CV)\), \((CM^*)\), and \((I)\), every instance of each schema

\[
\neg (A > \bot), \text{ whenever } \neg \neg A \text{ and } A \in L_0
\]

and that is closed under \(\mathcal{M}\), (RCMf) and (RCEA).

In the following we will call \textit{epistemic} any conditional system that satisfies the following constraint: the system should be characterizable by our epistemic model suplemented by a notion of change compatible with AGM.

\(^{11}\)\(\mathcal{E}\mathcal{F}\) does not belong to the family of counterfactual systems considered by David Lewis in [32]. Nevertheless Lewis’ semantics can be extended to validate \((SU)\). See Appendix C.
3.1 Ontic and opinionated conditionals

We will devote this section to studying a particular class of epistemic models that we will call opinionated. An opinionated model can be obtained from our standard epistemic models by introducing the following two restrictions: (O1) For every $(K, *, s)$, every consistent $K \in K$ is maximal, and (O2) the revision of every maximal theory in $K$ is also maximal. The opinionated models epistemically characterize the flat counterpart of Robert Stalnaker’s axiomatization of conditionals in [39]. Stalnaker’s system can be obtained from the axiomatic base that we proposed for the system $\mathcal{VW}$ by adding the axioms:

\[(A S) \quad A \land B \to A > B\]

\[(S) \quad (A > B) \lor (A > \neg B)\]

Now, the postulate $(K^\ast 4)$ validates $(A S)$ in the presence of restrictions $(O1)$-$(O2)$ imposed on opinionated models and $(S)$ is directly validated by $(O2)$.

Lindström and Rabinowicz [33] have made a distinction between ontic and epistemic conditionals, that seems akin to the one that we have just made between opinionated and non-opinionated conditionals. In their own words:

The epistemic conditionals have to do with hypothetical modifications of our beliefs about the world, while the ontic conditionals represent the hypothesis concerning what would be the case if the world itself were different.

It is tempting to conjecture that Lindström and Rabinowicz’s dichotomy can be captured from within the epistemic approach by representing the ontic conditionals as opinionated (epistemic) conditionals.\footnote{Hanson [22] seems to adopt this position.} Nevertheless some of the results proved in this section seem to restrict the universality of such connection. For the epistemic models are capable of representing only some of the systems of ontic conditionals. In fact, our epistemic models are not able to reconstruct Lewis’ “official” axiomatization of ontic conditionals: the system $\mathcal{VW}$, which can be axiomatized by taking $\mathcal{VW}$’s axiomatic base plus the axiom $(A S)$. In order to see this point it is enough to prove the following.

**Observation 3.7** $(A S)$ is not $e$-valid in any $EM$ satisfying $(K^\ast 3)$.

Proof: See Appendix B.

**Remark on nonmonotonicity**

It is clear that the following postulate is not satisfied by AGM:

\[(M^\ast) \quad A \to B \in Cn(\emptyset), \text{ then } K^\ast B \subseteq K^\ast A.\]
Moreover any notion of change that satisfies (M*) will not be compatible with AGM. Therefore, any conditional that satisfies the monotonicity postulate

(M>*) If \( A \rightarrow B \in \text{Cn}(\emptyset) \), then if \( B > C \), then \( A > C \).

will not be epistemic, i.e. all epistemic conditionals are non-monotonic.\(^{13}\)

### 3.2 Epistemic entailment

The postulate (K*4) does not seem to have any reflection at the flat conditional level – the notion of change needed to generate \( \mathcal{E} \mathcal{F} \) coincides with AGM minus (K*4). Nevertheless, (K*4) and (wp) have an impact at the flat level by increasing the set of positive patterns of inference.

**Observation 3.8** All \( f \)-instances of \( (A \land B) \models (A > B) \) hold in an EM \( \mathcal{M} \) iff \( \mathcal{M} \) satisfies (wp).

**Observation 3.9** All \( f \)-instances of \( A \rightarrow B, \neg(T \rightarrow \neg A) \models (A > B) \) hold in an EM \( \mathcal{M} \) iff \( \mathcal{M} \) satisfies (K*4).

Observations 3.7 and 3.8 indicate that a pattern \( \alpha \models \beta \) cannot be represented at the level of the object language by the PV of \( \alpha \rightarrow \beta \). In section 8 we will suggest that \( \alpha \models \beta \) be represented by \( \models \alpha > \beta \).

Nevertheless \( \gamma, \alpha \models \beta \) can be represented by \( \gamma \models \alpha \rightarrow \beta \) when \( \alpha, \beta, \gamma \) satisfy certain constraints.

**Observation 3.10** \( \sigma_1(A_1 > B_1), \ldots, \sigma_i(A_i > B_i) \models \sigma_{i+1}(A_{i+1} > B_{i+1}) \rightarrow (\sigma_{i+2}(A_{i+2} > B_{i+2}) \rightarrow (\ldots \rightarrow (\sigma_n(A_n > B_n) \rightarrow (A > B) \ldots))) \) iff \( \sigma_1(A_1 > B_1), \ldots, \sigma_{i+1}(A_{i+1} > B_{i+1}) \models \sigma_{i+2}(A_{i+2} > B_{i+2}) \rightarrow (\ldots \rightarrow (\sigma_n(A_n > B_n) \rightarrow (A > B)) \ldots) \), where \( \sigma_n \) is either empty or a negation, and \( A_i, B_i, A, B \) belong to \( L_0 \).

Proof: See Appendix B.

### 4 Positive versus negative validity

#### 4.1 Gärdensfors’ belief revision models

The initiated reader has probably noticed that the semantic system proposed in section 3 does not follow the classical Gärdensforsian proposal (see [14], chapter 7). In fact, not only does our epistemic model differ from that of Gärdensfors

\(^{13}\)Of course the class of nonmonotonic conditionals is larger than the class of epistemic conditionals. In fact, there are notions of change that are incompatible with AGM (like update) that do not satisfy (M*).
with respect to the fact that (SRT) is used instead of (GRT), but our definitions of epistemic entailment and validity are also very different from those of Gärdenfors. In this section the two semantic systems will be compared. Further results about the structure of the notion of validity introduced in section 3 will be also shown in the light of this comparison.

Before presenting Gärdenfors’ models we need to introduce some previous clarifications.

Let \( C \) be a finitary, consistent, compact, and superclassical \(^{14}\) notion of consequence defined over the language \( L_\succ \). In addition \( C \) is such that for all \( A, B \in L_\succ \) and \( K \subseteq L_\succ \); \( A \rightarrow B \) belongs to \( C(K) \) iff \( B \in C(\{K \cup \{A\}\}) \).

**Definition 3.0.2** A *conditional belief set* is a subset \( K \) of \( L_\succ \) which satisfies the following conditions: for every \( A, B \in L_\succ \), (1) \( K \) is non-empty, (2) If \( A \in K \) and \( B \in K \), then \((A \land B) \in K \), and (3) If \( A \in K \), and \( A \rightarrow B \in C(\emptyset) \), then \( B \in K \).

Gärdenfors considers *belief revision models* \( (K, *) \) where \( K \) is a set of conditional belief sets expressed in \( L_\succ \) closed under expansions and revisions, and \( * \) is a belief revision function \( *: K \times L_\succ \rightarrow K \), obeying (GRT). Gärdenfors’ notion of satisfaction is formulated as follows: an \( L_\succ \)-sentence is satisfiable in a model \( (K, *) \) iff \( A \) belongs to some consistent \( K \in K \).

According to Gärdenfors’ definition of validity a \( L_\succ \)-sentence \( A \) is epistemically valid iff its negation is not satisfiable in a system \( (K, *) \). We can call this notion of validity *negative validity* (NV hereinafter).

Gärdenfors \(^{12}\) claimed that his models constrained by \( B \) (\( B \) is defined in section 2) are able to establish the negative validity of \( VC \).\(^{15}\) Gärdenfors never explored in an explicit way the positively valid extensions of his models.\(^{16}\)

Of course in possible worlds semantics NV and PV coincide. But in the epistemic framework, where both inconsistent and incomplete belief sets are allowed in the models, we should expect that NV and PV come apart.

So, two questions arise immediately. Are there conditions on EMs under which PV and NV come apart? Are there conditions on Gärdenfors’ models that distinguish PV from NV?

\(^{14}\)In the sense that \( C(\emptyset) \) contains all classical tautologies and all their substitution instances in the language \( L_\succ \).

\(^{15}\)Strictly speaking Gärdenfors used a notion of change stronger than \( B \) that is presented in more detail in section 4.2. Nevertheless \( B \) is enough to generate \( VC \).

\(^{16}\)A different proposal is presented in \(^{14}\). The notion of change adopted in \(^{14}\) uses \( K^*6 \) instead of \( (sco) \); and adds \( (K5) \). In \(^{9}\) Fuhrmann showed that \( (sco) \) cannot be replaced by \( (K^*6) \), and in \(^{2}\) we showed that \( (K5) \) cannot be added because it conflicts with the Ramsey test. Moreover in \(^{2}\) we also proved that the milder idea of adding \( (K^*6) \) instead of \( (K5) \) generates nested conditionals not validated by \( VC \). So, we can conclude that the best model of \( VC \) offered by Gärdenfors is the one presented in \(^{12}\).
Our models can be extended with the following notions of validity, consistency and entailment.

**Definition 4.1.1** An $L>_{\omega}$ sentence $A$ is negatively valid (NV) in a $\langle K, *, s \rangle$ if $\neg A \not\in s(K)$ for every consistent $K \in K$.

**Definition 4.1.2** An $L>_{\omega}$ sentence $A$ is $e^-$-valid if it is NV in every model.

**Definition 4.1.3** For every expression $A_0, A_1, \ldots, A_n, A$ of $L>_{\omega}$: $A_0, A_1, \ldots, A_n \models e^{-} A$ iff it is not the case that there exists $K$, such that $A_0 \in s(K), A_1 \in s(K), \ldots, A_n \in s(K)$, and $\neg A \in s(K)$.

**Definition 4.1.4** An $L>_{\omega}$ sentence $A$ is positively consistent (PC) iff $\not\models e_{+} \neg A$.

**Definition 4.1.5** An $L>_{\omega}$ sentence $A$ is negatively consistent (NC) iff $\not\models e_{-} \neg A$.

NV and PV come apart in the context of our models.\(^{17}\)

**Observation 4.1** $(AS)$ is NV in an epistemic model $M$ if $M$ is constrained by $(wp)$.

Proof: See proof of Observation 2.2 of [4].

Observation 3.7 together with Observation 4.1 are enough to show that PV and NV do not coincide in the context of our models. Notice that the notions of PC and NC also come apart in the EMs. Of course any sentence that is NC is PC, but not every sentence that is PC is NC. For example the negation of $(AS)$ is PC but it fails to be NC.

In the last section we claimed that certain systems of ontic conditionals cannot be represented as systems of epistemic conditionals. Our example was Lewis’ system $\mathcal{VC}$. Nevertheless, in the last section we appealed exclusively to PV, and we did not consider the structure of the conditional systems to be negatively validated by our models – remember that Gärdenfors extended his belief revision models with a notion of NV in order to obtain $\mathcal{VC}$. Is it possible to generate the flat fragment of $\mathcal{VC}$ by lowering the standard of epistemic validity and without appealing to opinionation?

In order to show that the set of flat formulas negatively validated by the EMs coincide with the flat counterpart of $\mathcal{VC}$, we need to show at least that the EMs negatively validate every instance of (ID), (CC), (CC'), (MP), (CV), (I) and (AS), and that they guarantee closure under (RCEA) and (RCMf) as well as closure under standard modus ponens.

\(^{17}\)The problem of the contrast between NV and PV was treated at length in [4]. From now on we will refer the reader to proofs presented there.
Arló Costa and Levi [4] showed that although our models negatively validate all the above axioms as well as the rules (RCEA) and (RCMf), they do not guarantee closure under standard modus ponens. In other words the rule

\[ (M_e) \text{ If } \models_\alpha \alpha \alpha \rightarrow \beta \text{, then } \models_\alpha \beta, \text{ for } \alpha, \beta \in FL_e. \]

fails to hold in the context of our models. Therefore although the epistemic models negatively validate the axiomatic base of \( FL(VC) \) proposed above, they do not guarantee the NV of the entire system \( VC \). For example the \( VC\)-thesis \( \neg(T \rightarrow \neg A) \rightarrow (T \rightarrow A) \) is not negatively validated by any EM as long as the model in question is constrained by the postulate \( K = K^*T \) (see [4] for a proof of this claim). Two important consequences can be extracted from the above analysis.

The first consequence is related to the status of NV in the context of our epistemic models. It should be clear from the above analysis that NV is not an interesting notion of validity from a formal point of view.\(^{18}\) It can also be argued that NV is less intuitive than PV. Therefore we recommend that NV be dismissed as a plausible standard for epistemic validity.

The second consequence is related to the status of \( VC \) as an epistemic system. The above results seem to indicate that the "ontic" conditionals of \( VC \) cannot be represented as epistemic conditionals.

### 4.2 Ontic conditionals and update

In section 4.1 we asked two questions: Are there conditions on EMs under which PV and NV come apart? Are there conditions on Gärdenfors' models that distinguish PV from NV? The first question has already been answered in the affirmative. We will devote the following paragraphs to answering the second question.

Gärdenfors' claim in [12] was that (GRT) together with the postulates of \( B \)\(^{19}\) and the postulate \( (K^*L) \)\(^{20}\) negatively validate \( VC \). Let us call \( G \) the notion \( B \) extended with \( (K^*L) \). Now we can reformulate our second question in a more precise manner. Are the postulates of \( G \) together with (GRT) strong enough to positively validate \( VC \)? Or better, does a BRS constrained by \( G \) exactly induce the positive validity of \( VC \)? And, if it does not, is it possible to find a notion of change that, when combined with (GRT), exactly generates \( VC \), i.e. the PV of \( VC \)? Fuhrmann [9] answered the last three questions positively. Arlo Costa and Levi [4] answered the last question positively and showed that a positive

\(^{18}\)Nevertheless, there are interesting fragments of the language \( FL_e \) for which the distinction between NV and PV is unimportant. Arlo Costa and Levi [4] proved that NV and PV coincide for Horn-flat closure conditions - see section 5.1 for a definition of Horn-flat and Generalized-Horn-flat closure conditions.

\(^{19}\)See the definition of \( B \) in section 2.

\(^{20}\)(\( K^*L \)) can be obtained from \( (K^8) \) by substituting \( \neg(A \rightarrow \neg B) \) for the antecedent of \( (K^8) \).
answer to the first two questions depends on the behaviour of expansion in the context of Gärdenfors’ models (see [4] for a detailed explanation of this point). The proposal in [4] was to supplement G with the following postulate:

(SC) If $\neq + \dashv A$, then $K^*A$ is consistent.

Let us reserve the name $B_{SC}$ for the notion of change $G$ extended with (SC). The claim in [4] is that $B_{SC}$ together with (GRT) manages to positively validate $\forall C$.

So, there is a notion of change that conjoined with (GRT) manages to positively validate $\forall C$. The reader could be puzzled about this result. In the preceding sections we showed that $\forall C$ is not an epistemic system. Now we are claiming that Gärdenfors’ models, which prima facie appear to be similarly motivated to ours, are able to positively validate $\forall C$ using a notion of change that seems compatible with AGM. Fortunately the following distinctions can dissolve the puzzle. First the notion of change used by Gärdenfors in order to validate $\forall C$ is not completely determined by the postulates of $B_{SC}$. In fact, as Gärdenfors proved in [14] (GRT) is a “creative” definition of “>” in terms of “*”, that generates properties of *. Gärdenfors proved that (GRT) generates:

(KM) If $K \subseteq H$, $K * A \subseteq H * A$.

Moreover in [2] we proved that (GRT) also generates:

(US) If K is inconsistent, then $K^*A$ is inconsistent.

Therefore the notion of change that induces the PV of $\forall C$ is $C_{VC} = B_{SC}$ plus (KM) and (US). The standard proof of Gärdenfors’ “triviality” result (see [14]) tells us that $C_{VC}$ is incompatible with AGM.

So, our first point is that the notion of change needed to generate $\forall C$ via (GRT) is not compatible with AGM. Secondly in [4] Arló Costa and Levi showed that the philosophical motivations of $C_{VC}$ and AGM are quite different. In fact, in [4] Arló Costa and Levi showed that $C_{VC}$ can be modeled by a variant of a notion of change recently proposed by Katsuno and Mendelzon in [23] under the name of update.\textsuperscript{21}

In agreement with our previous conclusions according to which $\forall C$ should be classified as an ontic system rather than an epistemic system, update seems ontologically motivated rather than epistemically motivated. While AGM models the process of changing the content of our beliefs following a change of belief, update and $C_{VC}$ formally reflect a process of adapting beliefs following a change in the world.\textsuperscript{22}

The suppositional versions of AGM and $C_{VC}$ used in the eval-

\textsuperscript{21}A similar result was proved by G. Grønbe in [19].

\textsuperscript{22}The differences between AGM and update were pointed out by Katsuno and Mendelzon in [23]. A similar idea was also independently suggested by J. Collins in [7]. Collins proposed in [7] a weaker notion of change than update, called imaging, which is also appropriate to formalize ontic conditionals.
uation of conditionals also differ. The suppositional version of AGM formalizes the change that follows the supposition that a new item of information has been learned, while the suppositional versions of update and \( C_{VC} \) represent the process of suppositionally adapting beliefs following the supposition that a certain change in the world has happened.

Gärdenfors' belief revision models and our epistemic models seem to be designed to capture conditionals of very different kinds. The belief revision models are better suited to deal with ontic conditionals while our models are especially constructed to deal with epistemic conditionals. Of course the belief revision models can be adapted to generate some flat systems that we characterized as epistemic. For example the class of belief revision systems that satisfy the postulates of \( C_{VC} \) minus the postulate (wp) induce the positive validity of the system \( \mathcal{VW} \). If we focus on the flat counterpart of \( \mathcal{VW} \), both our systems and those of Gärdenfors can generate Flat(\( \mathcal{VW} \)). Nevertheless we will argue in section 8 that this similarity is an illusion generated by the poverty of the language \( FL_\omega \) used in the comparison. In section 8 we will propose an extension of the EMs capable of dealing with the iterated conditionals expressible in \( L_\omega \), and we will show that the iterated version of some of the axioms of \( \mathcal{VW} \) fail to be PV in the context of our models, while they are positively validated by Gärdenfors' models.

In the following three sections we will consider an important application of the epistemic models by showing that the family of preferential nonmonotonic logics proposed in [25] by Krauss, Lehmann and Magidor can be represented as epistemic systems.

5 Rational entailment

5.1 The minimal conditional systems that mirror the KLM systems.

In little more than a decade an impressive amount of work has been devoted to the study of the process of “jumping to conclusions” or making “tentative guesses”. The aim was to account for the way that conclusions are or ought to be reached on the basis of information that does not deductively imply these conclusions and which lacks the important property of monotonicity possessed by deduction.

D. Gabbay and Y. Shoham independently proposed in [11] and [38] the systematic study of the different formalisms studied in the literature, via the investigation of the formal properties of the inference relation generated by the system, i.e. the relation between a set of propositions \( A \) and a proposition \( x \), in the case that the relevant procedure under study authorizes inference of \( x \) from \( A \). Gabbay’s approach was syntactic, while Shoham’s was semantic. Shoham’s
notion of preferential entailment was generalized and related via representation results, with extensions of Gabbay’s syntactic system, first by Makinson [34] and secondly by Krauss, Lehmann and Magidor [25]. The systems developed by Lehmann et al. are usually called the KLM systems in the literature.

The main formal property that differentiates the notion of consequence ‘\( \models \)’ studied by KLM and others from the classical notion, is that KLM’s notion of consequence is non-monotonic, i.e. that the following property is not obeyed by KLM’s ‘snakes’:

\[(M \models \ ) \text{ if } \models A \rightarrow B, \text{ then if } B \models C, \text{ then } A \models C.\]  

The strongest system of preferential reasoning studied by Lehmann and Magidor [26] is the system of Rational Logic \( \mathcal{R} \). \( \mathcal{R} \) can be syntactically characterized as follows:

\[
\begin{align*}
\text{(R)} & \quad A \models A \\
\text{(LLE)} & \quad \models A \leftrightarrow B, \ A \models C \\
& \quad B \models C
\end{align*}
\]

\[
\begin{align*}
\text{(RW)} & \quad \models A \rightarrow B, \ C \models A \\
& \quad C \models B
\end{align*}
\]

\[
\begin{align*}
\text{(CM)} & \quad \models A \rightarrow B, \ A \models C \\
& \quad A \land B \models C
\end{align*}
\]

\[
\begin{align*}
\text{(AND)} & \quad \models A \rightarrow B, \ A \models C \\
& \quad A \models B \land C
\end{align*}
\]

\[
\begin{align*}
\text{(OR)} & \quad A \models C, \ B \models C \\
& \quad A \lor B \models C
\end{align*}
\]

\[
\begin{align*}
\text{(RM)} & \quad \models A \rightarrow C, \ A \models \neg B \\
& \quad A \land B \models C
\end{align*}
\]

There are two other important preferential systems. The system \( \mathcal{P} \) and the system \( \mathcal{R} \mathcal{R} \). \( \mathcal{P} \) can be obtained from \( \mathcal{R} \) by dropping (RM). \( \mathcal{R} \mathcal{R} \) can also be obtained from \( \mathcal{R} \) by dropping (CM).

Several authors have noticed the striking similarities between the syntax of KLM systems and the syntax of the non-nested fragments of conditional formalisms already proposed in the literature by philosophers and computer scientists. KLM systems induce non-monotonic notions of consequence, while the conditional approach works with a binary connective \( A > B \), where occurrences of “\( > \)” are forbidden in both the antecedent and the consequent of \( A > B \). For example the conditional counterpart of the nonmonotonic principle

\[
\text{(AND) If } A \models B, \ A \models C, \text{ then } A \models (B \land C)
\]

\[\text{23Here } \models A \text{ abbreviates } A \in \text{Cn}(\emptyset)\]
is any f-instance of the conditional axiom (CC) considered above.

Recent work in the area of conditional logic has shown that the intuitive correlation between conditional logic and non-monotonic logic can be made precise by appealing to conditionals patterns of inference of the following shape: \( \sigma_i(A_1 > B_1),...,\sigma_i(A_n > B_n) \vdash (A > B) \), where \( \sigma_i \) is either empty or a negation, \( A_i, B_i, A, B \) belong to \( L_0 \) and \( \vdash \) is the classical notion of consequence enriched with the axioms of the conditional system ... .

When all the \( \sigma_i \) are empty the patterns just introduced are usually called Horn-flat, and when the \( \sigma_i \) are either empty or a negation they can be called Generalized-Horn-flat (GHF hereinafter).\(^4\)

Arló Costa and Shapiro [3] showed that the theses of \( \mathcal{R} \) can be encoded by the GHF-counterpart of several well-known conditional systems (see [3] for a precise definition of the GHF-counterpart of a conditional system). Among these Lewis’ systems \( \mathcal{V}, \mathcal{W} \) and the system \( \mathcal{EF} \) share the required conditional patterns.\(^5\) Now, it is important to emphasize that the minimal conditional system whose GHF-counterpart mirrors the theses of \( \mathcal{R} \) (the system \( \mathcal{V} \)) has a model-theoretical structure that does not validate (MP).\(^6\) In fact, \( \mathcal{V} \) does not validate (MP) independently of any linguistic consideration.

Arló Costa and Shapiro also showed that the minimal conditional logic whose GHF-counterpart coincides with the patterns of \( \mathcal{P} \) is Burgess’ system \( \mathcal{B} \). Burgess’ system can be obtained by subtracting the axioms (CV) and (MP) from the axiomatic base of \( \mathcal{W} \) offered in section 3. Finally the system \( \mathcal{RR} \) coincides with GHF(\( \mathcal{NP} \)), where \( \mathcal{NP} \) is the system proposed by J. Delgrande [8]. \( \mathcal{NP} \) can be characterized by the axioms (ID), (CV), and (CC), and the rules (RCM), (RCEA) and (I), presented in section 3. As in the case of \( \mathcal{V} \) neither \( \mathcal{NP} \) nor \( \mathcal{B} \) validate (MP).

One of the consequences of the results presented in [3] is that the task of providing epistemic semantics for preferential logics can be reduced to the task of providing epistemic semantics for its conditional GHF-counterparts. Another important consequence of this result is that the conditional counterparts of the preferential systems are “deontic” systems that do not obey (MP).

5.2 Snakes do not obey modus ponens

For the sake of clarity we will adopt the following terminology: we will call defeasible the conditionals that do not obey (MP), and nonmonotonic the conditionals that fail to obey the property of monotonicity.

All the minimal conditional systems whose GHF-counterparts coincide with

\(^4\)The terminology GHF is adopted in [3].
\(^5\)See Appendix C for a proof of this claim.
\(^6\)In [3] Arló Costa and Shapiro used possible worlds models instead of epistemic models in order to characterize \( \mathcal{V} \).
a preferential system are nonmonotonic and defeasible. As we explained above, this is so independently of the complexity of the underlying language. This indicates that the GHF-counterparts themselves are nonmonotonic and defeasible. On the other hand we established in section 3 that our epistemic analysis of conditionals forces us to accept only the failure of monotonicity, not that of (MP). In fact, our “paradigmatic” epistemic system, the system $\mathcal{E}\mathcal{F}$, does satisfy (MP). That means that in order to reconstruct the preferential systems as a particular kind of epistemic systems, some relevant parameters should be adjusted in the epistemic model offered in section 3. The question is: which parameter(s)?

A natural candidate is the notion of revision used in the model. In fact, we know from observation 3.6 that if the model is constrained by AGM, (MP) will be satisfied. More specifically the notion of change used in the model of preferential conditionals cannot contain the postulate (K*3).

In section 6.1 we will show that there is indeed a naturally motivated notion of change (compatible with AGM) which fails to satisfy (K*3) and which can be used in order to model the family of preferential conditionals epistemically.

6 An epistemic model for rational entailment

Arló Costa and Shapiro [3] have proved the following.

**Observation 6.0**

$$\sigma_1(A_1 > B_1), \ldots, \sigma_n(A_n > B_n) \vdash_{\mathcal{V}} (A > B) \text{ iff } \beta_1(A_1 \vdash B_1), \ldots, \beta_n(A_n \vdash B_n) \mathcal{R}\text{-entails } (A \vdash B),$$

where $\beta_i$ is either empty or yields $(A_i \vdash B_i)$.27

Similar theorems showing the coincidence of the theses of $\mathcal{R}\mathcal{R}$, and $\mathcal{P}$ with the theorems of the GHF-counterpart of $\mathcal{N}\mathcal{P}$ and $\mathcal{B}$ were also proven in [3].

In this section we will offer an epistemic model of the GHF-patterns that constitute the conditional counterpart of the KLM systems. Let $\sigma_1(A_1 > B_1), \ldots,$ $\sigma_n(A_n > B_n) \models_{\mathcal{V}} (A > B)$ be a GHF-positive pattern induced by the epistemic model of $\text{Flat}(\mathcal{V})$ [$\models_{\mathcal{N}\mathcal{P}}$ and $\models_{\mathcal{B}}$ are characterized in a similar way]. Then we can prove the following observations:

**Observation 6.1** $\sigma_1(A_1 > B_1), \ldots, \sigma_n(A_n > B_n) \vdash_{\mathcal{N}\mathcal{P}} (A > B) \text{ iff } \sigma_1(A_1 > B_1), \ldots, \sigma_n(A_n > B_n) \models_{\mathcal{N}\mathcal{P}} (A > B)$

**Observation 6.2** $\sigma_1(A_1 > B_1), \ldots, \sigma_n(A_n > B_n) \vdash_{\mathcal{V}} (A > B) \text{ iff } \sigma_1(A_1 > B_1), \ldots, \sigma_n(A_n > B_n) \models_{\mathcal{V}} (A > B)$

**Proof:** See Appendix D.

27See [25] for a precise definition of rational entailment.
Observation 6.3 \( \sigma_1(A_1 > B_1), ..., \sigma_n(A_n > B_n) \vdash_B (A > B) \) iff \( \sigma_1(A_1 > B_1), ..., \sigma_n(A_n > B_n) \models^{+\phi} (A > B) \)

Observations 6.0 and 6.2 indicate that the rational logic \( \mathcal{R} \) is characterizable as the set of \( \text{GPF} \)-positive patterns induced by the epistemic model of \( \text{Flat}(V) \). Therefore the positive entailment patterns of the epistemic model of \( \text{GPF}(V) \) provide an alternative semantical characterization of the system \( \mathcal{R} \) that is notably simpler than the standard one offered by Krauss-Lehmann-Magidor.\(^{28}\)

Another important spinoff from this analysis is to make it plain that AGM and the system \( \mathcal{R} \) are not two sides of the same coin. In fact, the notion of revision characterized by the postulates \( \text{(K*2)}, \text{(CM*)}, \text{(K*6)}, \text{(K*7')}, \) and \( \text{(K*8)} \) certainly does not coincide with AGM.\(^{20}\) In section 6.2 we will provide a model for this notion of change and we will compare it with AGM. More importantly we will suggest that the notion of change needed to generate \( \text{GPF}(V) \) exactly is not powerful enough to capture some intuitive features associated with preferential reasoning. A notion of change that reflects these features will be presented and its conditional counterpart will be studied. This will be done immediately in section 6.1.

6.1 Preferential revision

The epistemic intuition that underlies the use of preferential conditionals can be expressed using the stratified Ramsey tests.

(P) The preferential conditional \( (A > B) \vdash (A > B) \) is accepted with respect to the belief set \( K \) if and only if the set that contains as many as possible of the expectations associated with \( K \) as are compatible with \( A \), has \[ \text{has not} \] \( B \) as a member.

The change operation involved in (P) seems to be a two-step process. First the agent switches (suppositionally) to an expectation set that extends the current belief set \( K \). Then the following question is considered. If the expectation set obtained in the first step were the current belief set, which would be the structure of the minimal change needed to consistently accommodate the sentence \( A \) in it? If the agent determines that the suppositional change of the expectation set contains \( B \) [does not contain \( B \)], he will conclude that \( (A > B) \vdash (A > B) \) is preferentially supported by \( K \).

If we represent the preferential change of \( K \) with a sentence \( A \) as \( K \circ A \), (P) can be rewritten as follows:

\[ \text{(SRTp)} \quad A > B \in s(K) \iff B \in K \circ A. \]

\(^{28}\) Via a double mapping whose first step \( \vdash \) is guaranteed by \[ 3 \], and whose second step \( > - \) is established by our epistemic semantics.

\(^{20}\) As we explained in section 3, the epistemic model of \( \text{Flat}(V) \) is only constrained by the postulates: \( \text{(K*2)}, \text{(CM*)}, \text{(K*6)}, \text{(K*7')}, \) and \( \text{(K*8)} \).
(SNRTp) \( \neg (A \supset B) \in s(K) \iff B \not\in K \circ A. \)

In the following paragraphs we will provide a model for the change operation \( \circ \) involved in (P).

### 6.1.1 Grove models

Adam Grove [20] has proposed the following model for revision. Consider the set of all maximal and consistent extensions of \( L_0 \). Call this set \( M \). Now we can define \( /K/ \) as the set of all \( M \) such that \( K \subseteq w \). Conversely, for any set \( S \subseteq M \), let \( \text{Th}(S) \) be defined as \( \bigcap \{w \in S\} \). Then a system of spheres corresponding to \( /K/ \) is a collection \( S \) of subsets of \( M \) such that it satisfies the following conditions.

(S1) \( S \) is totally ordered by inclusion.

(S2) \( /K/ \) is the \( \subseteq \)-minimum of \( S \).

(S3) \( M \) is in \( S \).

(S4) If \( A \) is a sentence and there is any sphere in \( S \) intersecting \( /A/ \), then there is a smallest sphere \( S_A \) in \( S \) intersecting \( /A/ \).

The set \( C(A) = /A/ \cap S_A \) is the set of the closest elements in \( M \) to \( /K/ \) in which \( A \) is an element. Grove’s idea is to represent \( /K^\circ A/ \) by \( C(A) \). A representation result is proved in [20]. Other type of belief change functions can be

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30 The idea behind (SRTp) is similar to Gärdenfors and Makinson’s interpretation of preferential entailment in terms of expectation sets (see [18]). Nevertheless Gärdenfors and Makinson do not think it necessary to involve the current belief set \( K \) in the model of preferential revision. They only work with the set of expectations associated with \( K \) and the AGM operation defined on it.

Fuhrmann and Levi’s model in [10] is also closely connected with (SRTp). They proposed the following inductively extended version of Ramsey test.

(IERT) \( A \supset C \in s(K) \iff C \in [K^\circ A] \vdash B \), where \( \vdash \) indicates the inductive expansion of \( K^\circ A \) with a sentence \( B \) (see [10] for details).

The following modification of Fuhrmann et. al.’s test may be closer to our approach.

(IERT’) \( A \supset C \in s(K) \iff C \in [K^\circ] \supset A \), where \( K^\circ \) is the deductive closure of \( K \) and some sentence whose addition to \( K \) is justified via inductive inference.

The idea behind (IERT’) is as follows: first the agent \( X \) asks himself the following question: what set of sentences could be added justifiably by inductive inference to \( K \)? Once this set \( K^\circ \) is suppositionally determined, the second relevant question is: if \( K^\circ \) were the current corpus instead of \( K \), how could \( K^\circ \) be minimally changed to incorporate \( A \)? \( B \) could be or could not be a member of \( [K^\circ] \supset A \). If it is, \( (A \supset B) \) is a preferential conditional supported by \( K \). If it is not, \( \neg (A \supset B) \) is preferentially supported by \( K \).

\( K^\circ \supset A \) does not necessarily coincide with \( [K^\circ] \vdash B \). Nevertheless it is reasonable to conjecture that the model offered here instantiates the particular case when \( [K^\circ] \supset A = [K^\circ A] \vdash B \) (see [31], sections 5-6-10, for the conditions on the \( \vdash \) (and \( i \)) operation that make this coincidence possible).

31 When \( /A/ \cap S \) is empty, \( S_A = \cup S_i \) by definition.
also represented by the model proposed by Grove. For example the proposition corresponding to the contraction of K with A\textsuperscript{32} can be represented by /K/ \cup C(A), in the case when /A/ \cap S is nonempty. Expansions are degenerate cases of revisions. In fact, for every proposition /A/ compatible with /K/, /K+A/ is represented as /K/ \cap /A/.

6.1.2 Preferential models

Consistent expansions of /K/ can be represented as subsets of /K/. We suppose that there is a function that yields a system of spheres corresponding to S for every non-empty S \subseteq M. Therefore, given a /K/ and a non-empty set of possible worlds S included in /K/, there is a system of spheres that corresponds to S.

The two-step process involved in (P) can now be represented as follows. First X suppositionally switches from K to Th(S), for some S \subseteq /K/. Secondly X uses the system of spheres associated with Th(S) to compute Th(S)*A. Then, if we represent the preferential change of K with A as K\circ A, K\circ A = Th(S)*A.

Th(S) and K are interrelated (K \subseteq Th(S)), but in general Th(S) is different from K. Moreover there is no necessary connection between the system of spheres associated with K and the system associated with Th(S). So, we should expect that preferential and standard revision were characterized by a different set of postulates. That this is the case will be verified with the help of the following model.

Let a default system of spheres corresponding to /K/ be a collection S of subsets of M such that it satisfies the following conditions:

(S1) S is totally ordered by inclusion.

(S2p) S contains a \subseteq-minimum set n(k), and n(k) \subseteq /K/.

(S3) M is in S.

(S4) If A is a sentence and there is any sphere in S intersecting /A/, then there is a smallest sphere S_A in S intersecting /A/.

Now the set C(A) = /A/ \cap S_A is the set of closest elements in M to n(K) in which A is an element. With these tools we can define the new notion of preferential revision “\circ”, as follows: /K\circ A/ = C(A)\textsuperscript{32} K\circ A now models the minimal change of the “default set” Th(n(K)) needed to accommodate A consistently in it.

Notice that K\circ T (where T is a tautology) has a particular meaning. In fact, it denotes the strongest “default-extension” of K. In other words, K\circ T denotes the set of expectations associated with K, i.e. Th(n(K)).

\textsuperscript{32}See [14] for a characterization of the contraction operation using rationality postulates
\textsuperscript{33} As before when /A/ \cap S is empty, S_A = \cup S_i by definition.
Preferential revision is not constrained by the postulate \( K \circ T \subseteq K \). The justification for this failure is as follows. The endorsement of \( K \circ T \subseteq K \) under the above interpretation of \( \circ \) amounts to assert that everything that is expected is believed true, which is false. Nevertheless the postulate \( K \subseteq K \circ T \) still holds. The reason is that under our construction of the notion of expectation, everything that is believed true should be expected.

In AI it is quite customary to read \( T \vdash A \) as “it is normal that \( A \)”, or as “it is expected that \( A \)”. Notice that our interpretation of the meaning of \( K \circ T \) is consistent with (SRTp) and this reading of \( T \vdash A \). In fact, to say that \( A \in K \circ T \) means, according to our construction, that \( A \) is expected, which, using conditionals, can be encoded by the formula \( T > A \).

Preferential revision satisfies all the postulates needed to generate \( \text{G}HF(\nu) \).

**Observation 6.4** Preferential revision satisfies the postulates \( (K\circ 2), (CMo), (K\circ 6), (K\circ 7'), (K\circ 8) \), \( (\text{AND}o) \) and \( (\text{RW}o) \).

The postulates \( (K\circ 2), (CMo), (K\circ 6), (K\circ 7') \) and \( (K\circ 8) \) can be obtained from \( (K^*2), (CM^*), (K^*6), (K^*7') \) and \( (K^*8) \) by substituting \( o \) for every occurrence of \( * \). The postulates \( (\text{AND}o) \) and \( (\text{RW}o) \) are as follows.

1. \( (\text{AND}o) \) If \( A \in K \circ A \) and \( B \in K \circ A \), then \( A \land B \in K \circ A \).
2. \( (\text{RW}o) \) If \( A \rightarrow B \in \text{Cn}(\emptyset) \), and \( C \in K^*A \), then \( C \in K^*A \).

Of course the notion of change used in every EM satisfies \( (\text{AND}o) \) and \( (\text{RW}o) \). In fact, this notion of change satisfies (by definition of EM) the postulate \( (K\circ 1) \) and every notion of change that satisfies \( (K\circ 1) \) satisfies \( (\text{AND}o) \) and \( (\text{RW}o) \).

The postulates mentioned in observation 6.4 are not strong enough to characterize preferential revision. In fact

**Observation 6.5** Preferential revision satisfies \( (K\circ 5) \): \( K \circ A \) is consistent if \( A \) is consistent, as well as: \( K \subseteq K \circ T \).

Preferential revision does not satisfy the offending postulate \( (K\circ 3) \): \( K \circ A \subseteq K + A \).\(^{34}\) Moreover the following observation can be established:

**Observation 6.6** The postulate \( (K\circ 4) \): If \( A \not\in K \), then \( K + A \subseteq K \circ A \), is not satisfied by preferential revision.

Nevertheless the following “shadowed” versions of \( (K\circ 3) \) and \( (K\circ 4) \) are satisfied:

\(^{34}\)To see that \( (K\circ 3) \) is not satisfied it is enough to consider the following instance of \( (K\circ 3) \): \( K \circ T \subseteq K \).
(K\textcircled{3}) \ K \circ A \subseteq (K \circ T) + A

(K\textcircled{4}) \text{ If } A \not\subseteq K \circ T, \text{ then } (K \circ T) + A \subseteq K \circ A.

The fact that (K\textcircled{3}) and (K\textcircled{4}) are not validated by \circ clearly establishes that \circ is weaker than AGM. At the same time preferential revision is stronger than the notion of change that exactly mirrors KLM's version of (rational) preferential reasoning. In fact, neither (K\textcircled{5}) nor K \subseteq K \circ T are needed in order to obtain GHF(\mathcal{V}) and both are satisfied by \circ.

The conditional correlate of the failure of (K\textcircled{3}) is the fact that (MP) is not validated by preferential revision together with the stratified versions of Ramsey test. The following example will help us to provide some intuitive support for this feature of the preferential conditionals induced by \circ via (SRTp).

Let K be Cn(Raven (Pete)), and Th(n(K)) be the extension of K obtained by expanding K with the expectation: Raven(Pete) \rightarrow Fly(Pete). In a situation of this sort our epistemic agent X will accept Raven(Pete) \rightarrow Fly(Pete)\textsuperscript{35}. In fact, the minimal AGM-change of Th(n(K)) needed to accommodate Raven(Pete) contains Fly(Pete).

Will X in this situation believe Fly(Pete)? No, X will only expect Fly(Pete). In other words, (MP) is violated - X accepts Raven(Pete) \rightarrow Fly(Pete) and X believes Raven(Pete), but X does not believe Fly(Pete). Nevertheless the example suggests that preferential reasoning sanctions the following weaker form of modus ponens:

(DMP) \text{ If } A \text{ is believed by } X \text{ and } A \rightarrow B \text{ is accepted by } X \text{ then } X \text{ expects } B.\textsuperscript{36}

The KLM systems do not have enough expressive power to represent a principle like (DMP).\textsuperscript{37} Does our model have enough strength to represent it? In order to consider this point let us first translate (DMP) to our formalism. Since

\textsuperscript{35}Where '$\rightarrow$' is our 'preferential' conditional.

\textsuperscript{36}The initials DMP stand for "deontic modus ponens". The terminology is borrowed from Scott Shapiro, who first suggested to me the importance of this principle in the area of the formalization of conditional obligation.

\textsuperscript{37}The system offered by J. Delgrande in [8] does not satisfy (DMP) either. But in the case of Delgrande's system, (DMP) cannot be added without violating some basic principles defended by Delgrande. In order to see why, we need to review first some of the ideas suggested by Delgrande in [8]. According to Delgrande the preferential conditional must satisfy the following constraint. It has to be capable of allowing for formulas of the following shape to be acceptable, while having true antecedents:

1. Raven (Pete) \rightarrow Fly (Pete)
2. Raven (Pete) \land Has a broken wing (Pete) \rightarrow not Fly (Pete)

Obviously any connective satisfying this constraint does not satisfy (MP).

It is important to notice, Nevertheless, that Delgrande’s constraint is quite strong, and that not every defeasible connective satisfies it. For example, the connective induced by our
we proposed representing the set of expectations associated with the current belief set $K$ by the "default set" $\text{Th}(n(K)) = K \circ T$, the epistemic counterpart of $(\text{DMP})$ should be written as follows:

$$(\text{DMP}^\circ) \text{ If } A \in K \text{ and } B \in K \circ A \text{ then } B \in K \circ T$$

$(\text{DMP}^\circ)$ is indeed satisfied by preferential revision $(\text{DMP}^\circ)$ and $K \subseteq K \circ T$ are equivalent in the presence of $(K \circ 3s)$. It is interesting to point out that $(\text{DMP}^\circ)$ does not induce the positive validity of any conditional formula. Nevertheless $(\text{DMP}^\circ)$ has an impact on the behavior of $\models^+$:

**Observation 6.7** All $f$-instances of the pattern $(\text{DMP}>)$: $A, B \models^+ T > B$ hold in a EM $M$ iff $M$ satisfies $(\text{DMP}^\circ)$.

In addition the postulate $(K \circ 5)$ - if $A$ is consistent then $K \circ A$ is consistent - induces $(\text{SU})$, which generates GHF patterns not included in GHF($\mathcal{V}$). In fact, $(\text{SU})$ entails $\neg(T > \bot)$, which is not a GHF($\mathcal{V}$)-thesis. So the conditionals induced by preferential revision are stronger than the conditional counterpart of the rational snakes proposed by KLM.

We will close this section with a final comparison between Gärdenfors' epistemic model and our model. It is interesting to notice that Gärdenfors' belief revision systems cannot be used to model the conditionals generated by preferential revision. In fact the following two facts can be proved.

**Observation 6.8** Every instance of $\neg (T > \neg A) \rightarrow (T > A)$, where $T$ is a tautology, is NV in Gärdenfors' belief revision models iff the model satisfies $K \subseteq K \circ T$ and for every $K \subseteq K$ and $A \in L^>_T$, $K+A = K \cup \{A\}$.

**Observation 6.9** $\neg (T > \neg A) \rightarrow (T > A)$ is not the conditional counterpart of any thesis of any preferential system.

Proof: We will verify that $\neg (T > \neg A) \rightarrow (T > A)$ is not a thesis of GHF($\mathcal{V}$). Consider a belief set $K$ such that neither $A$ nor $\neg A$ belong to $K \circ T$. Then $\neg (T > A)$ is supported by $K$, and $\neg (T > \neg A)$ is supported by $K$. Suppose now for contradiction that $\neg (T > \neg A) \rightarrow (T > A)$ is PV. Then $(T > A)$ should be supported by $K$. Contradiction.

Notion of preferential revision (which is constrained by $(\text{DMP})$) does not satisfy Delgrande's constraint. In fact, if $X$ accepts (1) and (2) plus Raven (Pete) $\wedge$ Has a broken wing (Pete), $X$ will also expect:

(3) Fly(Pete)
(4) It is not the case that Fly(Pete)

Let us also say that $X$'s belief set $K$ only contains the consequences of Raven (Pete) $\wedge$ Has a broken wing (Pete). Now (3) and (4) can only be expected in the case that the set of expectations associated with $K$ is inconsistent, a case ruled out by our semantics when $K$ is consistent. In other words, our preferential conditionals do not obey Delgrande's constraint.
6.2 Normal revision

In the previous section we verified that preferential revision is weaker than AGM. We also verified that this notion is, nevertheless, too strong to generate exactly the conditionals that mirror KLM’s rational snakes. In fact, our default system of spheres validates the postulate (Kσ5) which, in turn generates non-GHF( ≤ ) theses. Since (Kσ5) is semantically mirrored by the condition (S3) used in the definition of σ, there is an obvious modification of the default system of spheres capable of defining a notion of change that exactly induces GHF( ≤ ). The modification is as follows: delete condition (S3) in the definition of the default system of spheres and maintain the other conditions unchanged. This notion of change is adequate, but it is still unnecessarily strong. In fact, consider clause (S2p). (S2p) can be replaced with the following simpler condition without any loss of semantic power in order to obtain GHF( ≤ ).

(S2n) S contains a ⊆-minimum set n(k).

(S2p) is responsible for the validation of the postulates K ⊆ KσT and (DMP), but as we said in the above section, none of them positively validate flat or GHF-theses. These postulates augment the set of positive patterns of inference, without augmenting the set of positively valid theses.

Let us call normal revision the notion of revision characterized via a default system of spheres where condition (S3) is deleted and where condition (S2p) is replaced with condition (S2n). Let us also denote the normal revision of a set K with a sentence A by K • A. A notion of change intermediate between normal and preferential revision can be obtained by adding (S3) to the model of normal revision. This notion of change will be useful later. We propose to call it universal normal revision.38

As we argued above, normal revision is all that we need to generate the conditionals that mirror the rational snakes. Normal revision seems less epistemically motivated than preferential revision. At least, the set n(K) associated with K cannot be interpreted as a set of expectations associated with K, because this interpretation seems to demand the validation of the postulate K ⊆ KσT (if something is believed, then it must be expected). Perhaps normal revision can be seen under an ontological point of view. Suppose that the current set K does not represent our current beliefs but that it provides a description of the current situation or frame. Then n(K) can be seen as the most normal situation or frame envisaged jFrom K. If we interpret • under this point of view, then the postulate K ⊆ K • T is no longer needed (something abnormal can indeed be the case).

Discussions about normality have characteristically vacillated in the AI literature between epistemic and ontological constructions of the notion of normality.

38As we explain in section 8, condition (S3) is naturally related to the universal conditional systems of David Lewis.
Delgrande in [8], for example, defends a formalism that decidedly implements the ontological point of view, while discussing examples like the ornithological one considered in the above section, which seem ideal targets for the epistemic construction of normality. The intuitive appeal of deontic modus ponens to treat examples that supposedly illustrate "reasoning in normality conditions" seems a decisive factor in order to adjudicate in favour of the epistemic view encoded in preferential revision and against the ontological view promoted by normal revision.

**Remark on entailment patterns**

The contrast between normal and preferential revision shows that the conditionals validated by an epistemic model cannot exhaust all the semantic dimensions of the model in question. In fact, consider the following two notions of change: (a) normal revision and (b) a preferential notion of change that does not satisfy (K\textsubscript{5}). These two notions of change are quite different. The first admits an ontological interpretation, while the second seems epistemically motivated. Nevertheless, without appealing to entailment patterns we cannot tell the difference between these two notions of change at the syntactic level.

**Remark on nonmonotonicity and defeasibility**

Although the research on nonmonotonic logics has recently focused on nonmonotonic operators that in addition are defeasible, some researchers like Hirofumi Katsuno and Ken Satoh (see [24]) have insisted that (MP) probably plays an important role in the characterization of some kinds of nonmonotonic conditionals. According to the previous analysis an example of such conditionals seems to be provided by the operators that linguistically encode the commitments associated with our beliefs, rather than the commitments associated with our expectations.

7 **AGM and non-monotonic logic: two sides of different coins?**

In [17] Peter Gärdenfors and David Makinson investigated the close syntactical relationships between belief revision and nonmonotonic logic. In this section we will compare their results with ours.

Gärdenfors et al. used the following translation device to map snakes into stars:

\[(\models -\mathrm{RT}) B \in K \ast A \text{ iff } A \models_K B\]

In natural language:

A nonmonotonic inference of a proposition B from a proposition A is a discovery that B is contained in the result of revising a fixed back-
ground theory $K$ so as to integrate $A$. In this way, the nonmonotonic relation $A \vdash B$ serves as a shorthand for $A \vdash_K B$ which indicates that the nonmonotonic inference is dependent on the background theory $K$.

Notice that ($\vdash$-$\mathcal{RT}$) has to be supplemented with additional clauses in order to map certain AGM postulates (like ($K^3$) and ($K^4$)) which contain unrevised occurrences of $K$. In fact, in order to translate from * to $\vdash$, the following two (somewhat ‘ad hoc’) supplementary clauses are needed before applying ($\vdash$-$\mathcal{RT}$):

1. Inspect the condition on * to see whether it involves considering the revision of two theories $A$ and $A'$. If so, stop: the condition cannot in general be translated. If not continue.

2. Inspect the condition on * to see whether it involves reference to a theory in both its unrevised form $A$ and a revised form $A^*_x$ for one or more propositions $x$. If not, continue. If so, first break the condition up into a principal case that $A$ is consistent and a limiting case that $A$ is inconsistent. In the first case replace all ‘unrevised’ occurrences of $A$ by $A^*T$, where $T$ is an arbitrary chosen tautology. In the second case, eliminate all ‘unrevised’ occurrences of $A$ by treating elementary parts $x \in A$ of the condition to be translated as true.

Gärdenfors and Makinson’s idea is to show the existence of a syntactic correspondence between AGM and some notion of preferential consequence. When the procedure is applied to AGM the output is KLM’s system $\mathcal{R}$ supplemented by the snake-equivalent of our rule (SU) = $\mathcal{R}U$. In fact, when we apply the procedure to the conditions (R), (LLE), (RW), (CM), (AND), (OR), (SU) and (RM), we obtain the postulates (Ko2), (Ko6), (RWo), (CMo), (ANDo), (Ko7), (Ko5) and (Ko8) respectively. Therefore the output of the translation procedure coincides with the set of postulates that characterizes universal normal revision (see Observation 6.4).

Let $t(\text{AGM})$ be the notion of preferential logic obtained by applying Gärdenfors et al.’s translation procedure to AGM. It would be desirable to have $t(t(\text{AGM})) = \text{AGM}$. Nevertheless $t(t(\text{AGM}))$ yields a notion of change weaker than AGM (universal normal revision). In order to clarify this point let’s consider which is the exact impact of clause 2 in the construction of the output $t(\text{AGM})$. The application of step 2 to AGM leaves the postulates ($K^2$), ($K^5$), ($K^6$), ($K^7$) and ($K^8$) untouched. Postulates ($K^3$) and ($K^4$) are transformed into their “shadowed versions”. It can be verified that the postulates ($K^2$), ($K^5$), ($K^6$), ($K^7$) and ($K^8$) are equivalent to the postulates that characterize universal

\(^{39}\)Simply each elementary part of the form $x \vdash y$ should be replaced by $y \in K^*$, where $K$ is arbitrary but fixed background theory. Elementary parts of the shape $x \vdash y$ should be replaced by $y \not\in K^*$. 

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normal revision. In addition, the shadowed versions of \((K^*3)\) and \((K^*4)\) can be derived from universal normal revision. So, step 2 can be seen as a syntactic procedure by which the AGM conditions are translated into a weaker notion of revision that coincides with our notion of universal normal revision. After completing this intermediate step, the translation from universal normal revision to rational entailment is done via \((\vdash \text{-RT})\). In other words, adding the extra step 2 downgrades AGM to universal normal revision.\(^{10}\)

The following suggestion now seems natural: \((\vdash \text{-RT})\) and clause 1 are enough to provide a clean syntactical mapping between revision postulates and the system \(\mathcal{R}\). Of course this procedure will not map AGM into \(\mathcal{R}\), but universal normal revision into \(\mathcal{R}\).

Which is then the conditional counterpart of AGM? We have already answered this question in section 3, at least at the flat level; the system \(\hat{\mathcal{E}}\hat{\mathcal{F}}\). In section 8 we will verify in addition that the impact of the full AGM goes beyond the flat level. In fact, we will prove in the next section that some postulates, like \((K^*4)\), that do not increase the set of valid flat theses, validate iterated conditionals.

8 Iterated conditionals

The theory presented here has a crucial limitation: it does not provide acceptance conditions for iterated conditionals. To be more precise, the unmodified theory can only deal with certain types of nested conditionals. Consider, for example, an iterated conditional of the following shape: \(A > (B > C)\). It is clear that in this case the stratified Ramsey test recommends the acceptance of a conditional of this sort relative to a belief set \(\text{K}\) iff \(C\) is supported in \((K^*A)^*B\). But what can be done with conditionals like \((A > B) > C\)? A great deal of work has been done recently in order to answer that question.

An initial response was provided by Levi [29]. The idea is to extend the stratified test in order to cope with the problem of iteration, preserving at the same time the possibility of analysing flat conditionals with the full power of AGM. The extended test is as follows.

\textbf{(ISRT)} For any belief set \(\text{K}\) and any two sentences \(p, q \in L_\rightarrow\), \(p > q \in s(\text{K})\) iff \(q \in s(\text{K}')\) for any \(\text{K}'\) that is the outcome of a smallest possible transformation of \(\text{K}\) to make it support \(p\).

Of course, the problem of (ISRT) is to unpack “the outcome of a smallest possible transformation of \(\text{K}\) to make it support \(p\).” The problem is particularly complicated, because the transformation does not necessarily consist in a revision of \(\text{K}\) with some indicative sentence.

\(^{10}\)Notice that the postulate \(K = K^*1\) used in step 2 is not derivable as a theorem of universal normal revision.
Levi proposed in [31] a way of dealing with conditionals of the shape \( (A > B) > C \), when \( A, B \in L_0 \). Hannson proposed a different solution in [21] by postulating a primitive three-place relation \((K",K,K')\) between belief states with the intuitive meaning: "\(K"\) is at least as similar to \(K\) as is \(K'\)."

Two alternative solutions to the problem of iteration are provided in this volume, Boutilier and Goldszmidt (see [5]) tackle the problem by extending the AGM postulates in such a way that the extension deals with iterated revision. Lindström and Rabinowicz (see [33]) by contrast provide an indexical interpretation of conditionals that is also capable with treating iterated conditionals. Both constructions are compatible with an AGM analysis of flat conditionals.

We will not offer here an alternative solution to the ones already proposed. Rather, we will focus on the important consequences that a theory of epistemic iterated conditionals has for the correct understanding of the notion of epistemic entailment at the flat level. We will assume (ISRT) and a very mild constraint that is satisfied by all the theories of iteration mentioned above:

\[(C)\] If \( A \in L_\geq \) is supported by \(K\), then the minimal transformation of \(K\) needed to support \(A\) is \(K\) itself.

These elements are enough to establish the following representation result:

**Theorem 8.1** \(A \models B \iff A > B\), for \(A, B \in L_\geq\).

**Proof:** See Appendix E.

Theorem 8.1 can easily be generalized:

**Theorem 8.2** \(A_1, A_2, \ldots, A_n \models B \iff (A_1 \land A_2 \land \ldots \land A_n) > (A > B)\).

Theorem 8.1 tells us that the new notion of entailment \(\models\) can be represented at the level of the object language by the connective "\(>\)" similarly to the way that the classical notion of consequence \(\vdash\) can be represented by the material conditional "via" the deduction theorem.\(^4\)

Now we can easily determine the conditional formula validated by the preservation postulate \((K*4)\) - remember that \((K*4)\) does not generate positively valid flat formulas, but that it only has an impact on entailment patterns.

**Observation 8.1** All instances of \((\text{Pres})\): \(((A \rightarrow B) \land \neg(T > \neg A)) > (A > B)\) are \(PV\) in an EM iff the model satisfies \((K*4)\).

**Proof:** See Appendix E.

The formula \((\text{Pres})\) is validated by Lewis' system \(\forall C\), but it is not validated by weaker systems like \(\forall W\). This fact is surprising because we have already

\(^4\)One important difference between the two representation results is that the following is not true: \(A_1, A_2, \ldots, A_n \models B \iff A_1, A_2, \ldots, A_{n-1} \models A_n > B\).
verified that \( \mathcal{VC} \) is not an epistemic system. Moreover we can prove that any conditional system that contains (Pres) (M) and the iterated version of (MP) [(MPI) hereinafter], also contains the offending axiom (AS), that we proved to be positively invalid in section 3.

**Observation 8.2** (Pres), (MPI) and (M) entail (AS)

Proof: See Appendix E.

The fact proved in the above observation suggests that (MPI) is not PV. This is indeed the case:

**Observation 8.3** When \( B \in L_>, A \in L_0 \), and \( EM \) is constrained by \( (K^*) \), \( (A > B) \rightarrow (A \rightarrow B) \) is not e_+ -valid.

Proof: See Appendix E.

Observation 8.3 together with Observation 3.6, establishes that (MPI) is a theorem of epistemic conditionals only when the conditional language is restricted to its flat fragment. Once iteration is allowed, (MPI) is violated.

Van McGee [35] proposed the following intuitive scenario to illustrate failures of modus ponens like the one verified in Observation 8.1.

Opinions polls taken just before the 1980 election showed the Republican Ronald Reagan decisively ahead of the democrat Jimmy Carter, with the other Republican in the race, John Anderson, a distant third. Those apprised of the poll results believed, with good reason: If a Republican wins the election, then if it’s not Reagan who wins it will be Anderson. A Republican will win the election. Yet they did not have reason to believe If it’s not Reagan who wins, it will be Anderson.\(^{42}\)

As McGee pointed out in [35] the example can be generalized in order to give counterexamples to Stalnaker’s possible worlds semantics. Therefore we can conclude that: (1) the possible worlds approach and the epistemic approach differ even more dramatically at the iterated level than at the flat level, and (2) the epistemic approach deals comfortably with some of the counterexamples previously proposed against the possible worlds approach at the iterated level.

In [2] we proved (using Gärdenfors’ models) that the axiom \( (K^*5) \) has a nested conditional counterpart. Concretely \( (K^*5) \) (negatively) validates

\[
(T) \quad (\sim A > \perp) \rightarrow A
\]

\(^{42}\)Van McGee’s example can be modified in order to fit exactly the format of the proof of observation 8.1: If a Republican different from Anderson wins, then if it is not Anderson who wins, it will be a Republican. A Republican different from Anderson will win the election. Therefore, if it is not Anderson who wins, then it will be a Republican who wins.
\[(LU) \quad (\neg A > \bot) \rightarrow (\neg (\neg A > \bot) > \bot)\]

\[(UB) \quad A \rightarrow ((A > \bot) > \bot)\]

These theses also reflect \((K^*5)\) in the new semantic framework, when the model is constrained by the following postulate:

\[(K^*2n) \quad A \in s(K*A), \text{ for } A \in \mathbb{L}_.\]

**Theorem 8.3** All instances of the axioms \((T), (LU)\) and \((UB)\) are PV in an \(EM\), if the model satisfies \((K^*5)\), and \((K^*2n)\).

Now we have enough elements to determine which is the conditional counterpart of AGM. Such system can be obtained adding the axioms \((LU), (UB)\) and \((\text{Pres})\) to \(\mathcal{EF}\).

## 9 Summary

In this essay we have studied the logical structure of the conditionals that linguistically embody our *epistemic commitments*, i.e. our policies for belief change (at a certain instant \(t\)). We called these conditionals *epistemic conditionals*. The conditionals understood in this way do not express truth-value bearing propositions. The conditional “If \(A\), then \(B\)” just expresses linguistically the fact that if a rational agent supposes \(A\) true for the sake of the argument, then the corresponding suppositional state should contain \(B\). In other words, the conditional “If \(A\), then \(B\)” is accepted or supported relative to a belief state \(K\) of an agent \(X\), if and only if \(X\) is committed to accept \(B\) in the minimal suppositional change of \(K\) needed to accept \(A\) (SRT). By the same token, \(X\) accepts the negation of “If \(A\), then \(B\)” if and only if \(X\) is committed to reject \(B\) in the minimal suppositional change of \(X\)’s belief state needed to incorporate \(A\) (SNRT).

### 9.1 \(\mathcal{EF}\) and AGM: two sides of the same coin

Epistemic commitments can be constructed as functions that link a given epistemic state \(K\) and a suppositional item \(A\) with a uniquely determined suppositional state \(K*A\). In sections 2-4 we focused on the logical structure of the flat epistemic conditionals that arise when the function \(*\) is constrained by the postulates of AGM.

The notions of validity and entailment used in our models differ substantially from the ones used in [14] by Peter Gärdenfors. According to Gärdenfors, a sentence is considered valid in a system of belief sets if its negation is rejected in all consistent sets of the system (*negative validity*); while we considered valid only sentences that are accepted in all belief sets under consideration (*positive validity*). A notion of *epistemic entailment* was also introduced as follows: a
sentence A is positively entailed by a set Γ of sentences if and only if A is accepted in all belief sets where all the sentences of Γ are accepted.

Of course positive validity (PV) and negative validity (NV) coincide in possible worlds semantics. As Fuhrmann pointed out in [9] they can also coincide in the framework of Gärdenfors’ model for Lewis’ VC. But we showed in section 4 that PV and NV come apart in the context of our models.

We defended the conceptual primacy of the notion of PV, and therefore we considered that the flat conditional counterpart of AGM is formally captured by the system $\mathcal{E}_F$.

The preservation postulate (K*4) has always shown a peculiar behavior in the context of belief revision systems. It cannot be used together with Gärdenfors’ belief revision models on pain of inconsistency and although it is consistent with our models it does not seem to induce the PV of conditional theses at the flat level - although it does increase the set of positive inference patterns.

Does (K*4) have a conditional counterpart beyond the flat fragment of $L_>$ (i.e. does (K*4) induce the PV of a nested conditional schema?)? The answer is yes. In section 9.3 we will review the argument that led us to that conclusion.

9.2 Snakes and normal revision: two sides of the same coin.

In sections 5, 6 and 7 we provided an epistemic semantics for the most salient systems of nonmonotonic preferential logic. The exact notion of change (normal revision) needed to generate the strongest preferential system (R) was studied in detail in section 6. An important spin-off from this analysis was to provide an alternative semantic characterization of the preferential systems that is notably simpler than the one offered by Kraus-Lehmann-Magidor.

Our model for normal revision can be seen as a reconstruction of the epistemic assumptions that motivated the development of the preferential systems. We found some of these assumptions objectionable, and therefore a modification of the preferential systems based on an extension of normal revision, which we called preferential revision was offered.

To use the coin’s metaphors popularized by Gärdenfors in [16], AGM and the rational system $\mathcal{R}$ are two sides of different coins. The first coin has the nonmonotonic system $\mathcal{R}$ as one of its sides, or, if we want to use conditionals, the GHF fragment of the system $\mathcal{V}$. Then the flip side will coincide then with normal revision. The second coin has AGM as one of its sides, and its flat conditional counterpart is the system $\mathcal{E}_F$.

\textsuperscript{43}Gärdenfors first proposed in [16] the following question: “Are AGM and non-monotonic logic two sides of the same coin?”
9.3 Nested conditionals

Several authors have suggested (see [13] for example) that the ability to deal
with iteration is a necessary condition of adequacy that must be imposed on
any satisfactory theory of conditionals.

Many efforts have been devoted to extending the epistemic kernel in order to
provide acceptance conditions for nested conditionals. The research in the area
of iteration is still in a state of flux, with several interesting theories competing
for universal acceptance. Nevertheless all these theories share some basic con-
straints that we explicit under the form of rationality postulates in section
8. The postulates were used, in turn, to prove the following epistemic analogue
of the classical deduction theorem: an \( L_\omega \) sentence \( B \) is positively entailed by
another \( L_\omega \)-sentence \( A \) iff the conditional \( A \succ B \) is PV. This representation
result helped us to consider the question posed at the end of section 9.1: does
\( (K^*4) \) positively validate nested formulas? As we said above the answer is yes.
The postulate (Pres): \( (A \rightarrow B) \wedge (\neg (T \succ \neg A)) \succ (A \succ B) \), mirrors \( (K^*4) \) at
the iterated level.

In section 8 we also identified the iterated formulae that correspond to \( (K^*5) \)
(the theses (UB) and (LU)), and we proved the failure of the iterated version of
(MP). Therefore we concluded that the system \( \mathcal{E} \mathcal{F} \) plus the theses (Pres), (LU)
and (UB), constitutes the conditional counterpart of AGM.

9.4 Lapses of faith

Some objections traditionally raised against the so-called \textit{dispositional} theory of
conditionals\textsuperscript{44} can easily be adapted to attack the semantic framework presented
here.

Consider, for example, the two following epistemic conditionals.

1. If Kepler’s laws of motion are true, then I am the Pope.

2. If my theory of tides is false, then I am the Pope.

Galileo Galilei could have accepted both conditionals. In fact, Galileo a-
damantly defended the idea that the celestial motions were circular, and never
abandoned his theory of tides - a false theory that contradicted some of Galileo’s
own physical ideas, but which Galileo defended as a sort of \textit{idée fixe}.

The alleged problem with (1) and (2) is that our agent (let us call him G),
upon learning the antecedents of (1) and (2), will not be at all prepared to
accept its consequents, therefore violating the only-if part of the Ramsey test.
Once the faith is lost, apparently several examples of this type can be found, as
Lindström and Rabinowitz [33] pointed out.

\textsuperscript{44}Roughly the dispositional theory asserts that our rational \textit{dispositions} to change may be
identified with our conditional beliefs expressed in conditional sentences. One of the latest
defenses of the dispositional theory can be found in [36].
Do the kind of examples presented above threaten in any serious way the only-if part of the Ramsey test? We will suggest here that the answer to that question is no. Conditionals (1) and (2) linguistically represent the commitments acquired by our agent G at time t. Which is the content of these commitments? Presumably G thinks at t that the antecedents of (1) and (2) are not entertainable, i.e. that any suppositional change made at t to add them will end up generating a suppositional inconsistent state. The rationality of such commitments could be called into question, but even if we accept that G did have these commitments at t, this will not entail that at a later time t' G has to believe, against his deepest convictions, that he is the Pope upon learning the antecedents of (1) or (2). This would be true only if we supplement our theory with the following rationality constraint: suppositional commitments acquired at a time t should be tenaciously maintained over time. But this constraint does not seem reasonable. New evidence gathered at t' can persuade our agent to drop previous commitments acquired at t. My being committed to suppose B upon supposing that I learn A, does not entail that I will believe B upon learning A, but only that I will believe B upon learning A, while I have this commitment.

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\footnote{According to our rule (SU) the logical consistency of a certain view A is a sufficient condition for the entertainability of A. Therefore G (acting as a rational agent) should have considered the antecedents of (1) and (2) entertainable. It can be argued nevertheless that (SU) is too strict, and that it can be violated - Paracelsus' views, for example, are probably non-entertainable for a contemporary agent. But what about examples like \{Y\}: If Yeltsin hates Vatican, then I am the Pope! The antecedent of (Y) should be considered entertainable. But then what could be the justification for supposing the consequent of (Y) upon supposing that its antecedent is learned? Does a rational agent really have commitments of this sort? We can say no, and therefore we can dispose of the apparent counterexample by simply denying that a rational agent could have accepted (Y) in the first place. Of course this response begs the question somewhat, as Lindström and Rabinowicz pointed out in their analysis of a similar example in \cite{33}. We apparently appealed to the very same part of the test that we are defending in order to defend it. Is this “virtuous” circle satisfactory? It is quite tempting to answer in the affirmative. In any case, as we are about to show above, even when a rational agent can sometimes have commitments that “ex post facto” could seem unreasonable, this fact does not necessarily commit the agent to change her views according to these commitments upon learning new evidence.}
References


Appendix. Theorem section 3.

Theorem 3.1 A flat formula $A$ is e-valid iff $A$ is a theorem in $\text{Flat}(\mathcal{C}, \mathcal{M})$.

(Soundness) By construction all tautologies expressible in $L_0$ are included in all belief sets. Moreover, for all models $\langle \mathbf{K}, *, s \rangle$ all tautologies of $FL_>$ are included in $s(K)$ for all $K \in \mathbf{K}$ (by condition (3) on the support function $s$). Now let us consider the axiom $I$. Suppose for the sake of contradiction that there is $K \in \mathbf{K}$ for a model $\langle \mathbf{K}, *, s \rangle$, such that a $\xi$-instance $A > T$ of $I$ does not belong to $s(K)$. Then by (SRT) $T \not\in K^\star A$. But this is contradictory because $T$ belongs to every $K^\star A$ (every $K^\star A$ is a belief set). All $\xi$-instances of (CC) are PV. In fact, consider by the sake of contradiction that there is a model $\langle \mathbf{K}, *, s \rangle$ and a belief set $K \in \mathbf{K}$, such that $\not\models (A > B) \lor (A > C)$ does not belong to $s(K)$, for some $K \in \mathbf{K}$ in a model $\langle \mathbf{K}, *, s \rangle$ (with $\models (B \rightarrow C)$). Then $\not\models (A > B) \not\in s(K)$ and $(A > (B \land C)) \not\in s(K)$. Therefore, by (SRT) and (SNRT), $B \in K^\star A$, $C \in K^\star A$ and $(B \land C) \not\in K^\star A$.

Finally PV is preserved under $M$. In fact, assume $\models^+_\star \alpha$ and $\models^+_\star \alpha \rightarrow \beta$, for $\alpha, \beta \in FL_>$. Then by condition (3) on the support function $s$, $\models^+_\star \beta$.

To show the converse we will assume that a flat $\alpha$ is not a theorem in $\text{Flat}(\mathcal{C}, \mathcal{M})$, and we will show that $\alpha$ is not supported by some belief set $K$ of some epistemic model $\langle \mathbf{K}, *, s \rangle$. So, in the following we will construct explicitly a $\mathcal{CM}$-model $\langle \mathbf{K}, *, s \rangle$, and we will exhibit a $K \in \mathbf{K}$, such that $\alpha$ does not belong to $s(K)$.

Consider now any set $G_0$ expressed in $FL_>$ and the following consistent completion $G_0$.

Form a list of all formulae of $FL_>$ of the shape $\neg(A > B)$: $g_1, g_2, ..., g_{n\rightarrow}$. We suppose that each $FL_>$-sentence of the shape $\neg(A > B)$ (with $A, B \in L_0$) occurs at least once in this list. Now, with respect to this list, we construct an infinite sequence of sets

$I_0, I_1, ..., I_{n\rightarrow}$

as follows. As $I_0$ we take $G_0$ i.e.,

A set $\Sigma$ is $\mathcal{CM}$-inconsistent iff $\Sigma \vdash_{\text{Flat}(\mathcal{CM})} \gamma$, for all $\gamma \in FL_>$. Otherwise we say that $\Sigma$ is $\mathcal{CM}$-consistent. From now on we will use, for the sake of brevity, the terms "consistent" and "inconsistent" as abbreviations of the terms "$\mathcal{CM}$-consistent" and "$\mathcal{CM}$-inconsistent" respectively.
\[ I_0 = G_0 \] 
Then, for each positive integer \( n \) we set:

\[
I_{i+1} = \begin{cases} 
I_i, g_{i+1} & \text{if } I_i, g_{i+1} \text{ is consistent} \\
I_i & \text{otherwise}
\end{cases}
\]

We then form a Lindenbaum-type set: \( \overline{G_0} = C(\cup I_i) \), where \( \cup I_i \) denotes the union of all the infinitely many sets \( I_i \).

We will start our construction by defining the following function \( s \) for every set \( K \) expressible in \( L_0 \):

\[
s(K) = \begin{cases} 
\overline{\text{Flat}(C,M) \cup K} & \text{if } K \neq \text{CN}(\emptyset) \text{ and } K \text{ is consistent} \\
\overline{\text{Flat}(C,M)} & \text{if } K \neq \text{CN}(\emptyset) \text{ and } K \text{ is inconsistent} \\
\overline{\text{Flat}(C,M) \cup \left\{ \alpha \right\}} & \text{otherwise}
\end{cases}
\]

Since by hypothesis \( \alpha \) is not a theorem of \( \text{Flat}(C,M) \), and the completion of any consistent set is consistent, \( s(\text{CN}(\emptyset)) \) is consistent. Now the following series of sets of sets can be constructed:

\[
K_0 = \{ \text{CN}(\emptyset) \} \\
K_{n+1} = K_n \cup \{ C \in L_0 : A > C \in s(K) \text{ and } \neg (A > C) \notin s(K) \} \\
ak_1 \in K_n, A \in L_0 \} \cup \{ \text{CN} (K \cup \{A\}) : K' \in K_n, A \in L_0 \}
\]

We now have enough elements to construct our model \( (K, *, s) \). Take \( K = \cup K_i \) and for every \( K \in K \) and \( A \in L_0 \), define:

\[ K^*A = \{ C \in L_0 : A > C \in s(K) \text{ and } \neg (A > C) \notin s(K) \} \]

\( s \) was explicitly defined above. It is obvious that there is at least one belief set \( K \in K \), such that \( \alpha \notin s(K) \). In fact, in order to show this point it is enough to take \( K = \text{CN}(\emptyset) \).

Finally we have to verify that \( (K, *, s) \) is indeed a \( C,M \)-epistemic model. Definition 3.0.1. as well as our characterization of the notion of consequence \( C \) and the support function \( s \), guarantees that for every \( K \in K \), \( s(K) \) is a conditional support set. We need to check in addition conditions 1-3, and the positive and negative Ramsey tests. Conditions 1-3 are immediate.

If \( B \in K^*A \) by construction \( A > B \in s(K) \) and \( \neg (A > B) \notin s(K) \). This verifies one half of the positive Ramsey test and one half of the negative Ramsey test.

If \( B \notin K^*A \), then either \( A > B \notin s(K) \) or \( \neg (A > B) \notin s(K) \). There are three possible cases:

1. \( A > B \notin s(K) \) and \( \neg (A > B) \in s(K) \).
2. \( A > B \notin s(K) \) and \( \neg (A > B) \notin s(K) \).
3. \( A > B \in s(K) \) and \( \neg (A > B) \in s(K) \).

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Case (3) can be eliminated. In fact, it is easy to see that $s(K)$ is consistent if $K$ is, and both (SRT) and (SNRT) apply only to consistent belief sets. We can discard (2) as well. In fact, every $s(K)$ is maximal with respect to formulas of the shape $A > B$. Then (1) remains which is in agreement with both negative and positive Ramsey tests.

Finally we will check that all sets in $K$ are belief sets that support all theorems in $\text{Flat}(C_M)$. The definition of the support function trivially satisfies the requirement that all sets in $K$ support all theses in $\text{Flat}(C_M)$. It only remains to check whether all sets in $K$ are belief sets. We will verify by induction that the rationality constraints required in Definition 2.1 are satisfied. The basic case is obvious. Suppose that all sets in $K_n$ are belief sets supporting all theorems in $\text{Flat}(C_M)$. We will prove that all sets in $K_{n+1}$ are also belief sets. $K_0$ as well as the indicative expansions in $K_{n+1}$ are obviously belief sets. Now we have to show (for every $K' \in K_n$ and $A \in L_0$) that $K'^*A = \{ C \in L_0 : A > C \in s(K') \}$ and $\neg (A > C) \notin s(K')$, is a belief set. $K'^*A$ is non-empty, because by inductive hypothesis $A > T \in s(K')$. Assume now that $B \in K'^*A$, and $C \in K'^*A$. Then $A > B$ and $A > C$ belong to $s(K')$. By inductive hypothesis $s(K')$ contains all the theorems of $\text{Flat}(C_M)$, and therefore $(A > (B \land C)) \in s(K')$. In addition $\neg (A > (B \land C)) \notin s(K')$, taking into account the fact that $s(K')$ is, by construction, a consistent set. Therefore $B \land C \in K'^*A$. Finally assume that $C \rightarrow D$ is a truth functional tautology, where $C, D \in L_0$ and $C \in K'^*A$, with $A \in L_0$. Then $A > C \in s(K')$, and by (RCMf), $A > D \in s(K')$; which means that $D \in K'^*A$.

B Proofs of main observations. Section 3

Observation 3.1 A formula is derivable in $C_{M+}$ iff it is PV in the class of all EMs satisfying (K*6).

We will prove the following two lemmas:

**Lemma 1:** If a formula is derivable in $C_{M+}$, then it is PV in the class of all EMs satisfying (K*6).

Proof: Since we already proved the soundness of $\text{Flat}(C_M)$, we only need to prove the positive validity of every $\ell$-instance of each schema $(A > C) \leftrightarrow (B > C)$, whenever $\vdash A \leftrightarrow B$ and $A, B, C \in L_0$ for models constrained by (K*6). Assume $\vdash A \leftrightarrow B$, and $A, B, C \in L_0$. So, for all models $(K, \ast, s)$ that satisfy (K*6), $K^*A = K^*B$, for all $K \in K$. Assume now that one of these models $(K, \ast, s)$, is such that for some $K \in K$, $(\neg (A > C) \lor (B > C)) \notin s(K)$. Then $\neg (A > C) \notin s(K)$ and $(B > C) \notin s(K)$. But then by (SNRT) $C \in K^*A$. Since $K^*A = K^*B$, $C \in K^*B$, which in turn implies $(B > C) \in s(K)$. Contradiction. The same argument can be used to prove that $(B > C) \rightarrow (A > C) \in s(K)$ for
every $K \in K$ of every model $(K, s, s)$ constrained by $(K^*)$. Therefore every f-instance of each schema $((A > C) \leftrightarrow (B > C)) \in s(K)$, for every $K \in K$ of every $(K, s, s)$ constrained by $(K^*)$, whenever $\vdash A \leftrightarrow B$ and $A, B, C \in L_0$.

**Lemma 2.** Every EM in which (RCEA) is PV satisfies $(K^*)$. Assume that $\vdash A \leftrightarrow B$, with $A, B \in L_0$. Then for every $C \in L_0$ and for every $(K, s, s), ((A > C) \leftrightarrow (B > C)) \in s(K)$, for every $K \in K$. Assume $C \in K^*A$. Then $(A > C) \in s(K)$, which implies that $(B > C) \in s(K)$. By (SNRT) this generates $C \in K^*A$. The converse is immediate.

**Observation 3.3** Assume that all instances of (ID) are PV in $M$. Then all f-instances of (CV): $((A > C) \land \neg((A > \neg B)) \rightarrow ((A \land B) > C)$ are PV in $M$ if the model satisfies $(K^*)$.

**Proof:** Assume for the sake of contradiction that there is $K \in K$, such that: $((A > C) \land \neg((A > \neg B)) \rightarrow ((A \land B) > C) \notin s(K)$, with $A, B, C \in L_0$. Then $\neg((A > C) \notin s(K), (A > \neg B)) \notin s(K)$ and $((A \land B) > C) \notin s(K)$. By (SNRT) we have $C \in K^*A$. By (SRT) $\neg B \notin K^*A$ and finally by (SRT) $C \notin K^*(A \land B)$. Since $C \in K^*A$, $C \in (K^*A)+B$ also. But then, according to $(K^*)$, $C \in K^*(A \land B)$. Contradiction.

Assume that $\neg B \notin K^*A$ and that $C \in (K^*A)+B$. Then by (SNRT) $\neg(A > \neg B) \in s(K)$, $\neg(A > \neg B) \in s(K)$, $\neg B \in s(K)$, and $\neg B \in s(K)$. By (CV) $(A \land B) > B \rightarrow C \in s(K)$, from which we conclude, by (SRT) and the validity of (ID), that $(A \land B) > C \in s(K)$. Therefore $C \in K^*(A \land B)$.

**Observation 3.4** All f-instances of (CC): $((A > C) \land (B > C)) \rightarrow ((A \lor B) > C)$ are PV in an EM if the model satisfies the basic postulates and $(K^*)$: $K^*A \land K^*B \subseteq K^*(A \lor B)$.

**Proof:** Assume for the sake of contradiction that there is a consistent $K \in K$ such that $((A > C) \land (B > C)) \rightarrow ((A \lor B) > C) \notin s(K)$, with $A, B, C \in L_0$. Then $\neg(A > C) \notin s(K), \neg(B > C) \notin s(K)$, and $((A \lor B) > C) \notin s(K)$. By (SNRT) $C \in K^*A$, and $C \in K^*B$. Therefore by $(K^*) C \in K^*(A \lor B)$. Contradiction.

Assume now that (CC) is PV. Assume also that $C \in K^*A \land K^*B$. Then by (SRT) $A \lor C \in s(K)$ and $B \lor C \in s(K)$. By (CC) $(A \lor B) > C) \in s(K)$, and by (SRT) again $C \in K^*(A \lor B)$.

**Observation 3.6** All f-instances of (MP): $(A > B) \rightarrow (A \rightarrow B)$ are PV in an EM if the model satisfies $(K^*)$.

**Proof:** Assume for the sake of contradiction that there is $K \in K$ such that $(A > B) \rightarrow (A \rightarrow B) \notin s(K)$, with $A, B, C \in L_0$. Then $\neg(A > B) \notin s(K)$, and $(A \rightarrow B) \notin s(K)$. By (SNRT) $B \in K^*A$. By $(K^*) B \in K^*A$, from which we can conclude that $(A \rightarrow B) \in s(K)$. Contradiction.

Assume that (MP) is PV. Assume also that $C \in K^*A$. Then $A > C \in s(K)$. 41
Then immediately \((A \rightarrow B) \in s(K)\). By constraint (1) of the support function \((A \rightarrow B) \in K\). Therefore \(B \in K+\).

**Observation 3.7** (AS) is not PV in any EM constrained by \((K^*3)\).

Proof: Assume by contradiction that (AS) is e-valid. Take the following instance of it: \(A \rightarrow (T > A)\), with \(A \in L_0\). Consider any EM \((K,*,s)\), and \(K\in K\), such that \(\neg A \not\in K\) and \(A \not\in K\). By \((K^*3)\) \(K^*T\) is included in \(K+T = K\). Then since \(A \not\in K\), \(A \not\in K^*T\). But by the negative Ramsey test \(\neg(T > A)\) belongs to \(s(K)\). Since \(s(K)\) is closed under logical deduction \(\neg A \in K\).

**Observation 3.10** \(\sigma_i(A_1 > B_1),\ldots,\sigma_i(A_i > B_i) \models \sigma_i(A_{i+1} > B_{i+1}) \rightarrow (\sigma_i(A_{i+2} > B_{i+2}) \rightarrow (\ldots \rightarrow (\sigma_i(A_n > B_n) \rightarrow (A > B) \ldots)))\) iff \(\sigma_i(A_1 > B_1),\ldots,\sigma_i(A_{i+1} > B_{i+1}) \models \sigma_i(A_{i+2} > B_{i+2}) \rightarrow (\ldots \rightarrow (\sigma_i(A_n > B_n) \rightarrow (A > B) \ldots))\). Therefore there is a model \((K,*,s)\) and a \(K\in K\), such that \(\sigma_i(A_1 > B_1),\ldots,\sigma_i(A_i > B_i) \not\models (\sigma_i(A_{i+1} > B_{i+1}) \rightarrow (\sigma_i(A_{i+2} > B_{i+2}) \rightarrow (\ldots \rightarrow (\sigma_i(A_n > B_n) \rightarrow (A > B) \ldots)))\). By \((\text{SRT})\), \((\text{SNRT})\) and the assumption of the proof we have that \((\sigma_i(A_{i+2} > B_{i+2}) \rightarrow (\ldots \rightarrow (\sigma_i(A_n > B_n) \rightarrow (A > B) \ldots))) \in s(K)\); which indicates that \((A > B) \in s(K)\). Contradiction. The proof of the converse is similar.

### C Main proofs. Section 5

In this section we will prove that
\[
\text{GHF}(\mathcal{V}) \text{ coincides with } \text{GHF}(\mathcal{VW})
\]

We will prove the coincidence of those GHF fragments using the standard tools of possible-worlds semantics. A by product of this proof is to offer a possible-worlds characterization of the system \(\mathcal{E}\).

**Proof.** We will use a modified and extended version of Lewis’ semantic scheme\(^{48}\) in order to exhibit a proof. We will first introduce the necessary

---

\(^{44}\)Notice that \((\text{SRT})\) and \((\text{SNRT})\) guarantee that for all \(A, B \in L_0\) and \(K \subseteq L_0\), either \(\sigma(A > B) \in s(K)\) or \(\neg \sigma(A > B) \in s(K)\).

\(^{48}\)Modified because we are not using Lewis’ standard definition of valuation, and extended because we are working with conditions like \((SU)\) that were not investigated originally by.
semantical tools, and then we will immediately present the proof.

**Definition 1.1** A spheres model is double \( \langle I, \$ \rangle \) where \( I \) is a nonempty set of worlds, \( \$ \) a function that assigns to each world \( i \in I \), a nested set \( \$_i \) of subsets of \( I \) (called a system of spheres for \( i \)).

**Definition 1.2** A valuation for \( \langle I, \$ \rangle \) is a map \( P \) that assigns to each sentence a subset of worlds in \( I \). \( P \) is classical for the nonconditional fragment, and can be extended for the conditional fragment as follows:

\[ P(A > B) \text{ is the set of worlds of all } i \in I \text{ such that either:} \]

1) no \( A \)-world belongs to any sphere \( S \) in \( \$_i \), or
2) \( A \rightarrow B \) holds at every world in the smallest \( A \)-permitting sphere in \( \$_i \), or
3) There is at least an \( A \)-permitting sphere in \( \$_i \), but no smallest \( A \)-permitting sphere.

**Definition 1.3** For any given spheres model \( S \), \( \models_S (A > B) \) iff \( I = P(A > B) \).

A sentence schema is valid unconditionally if it is valid in every spheres model of the kind specified above. Nevertheless certain constraints can be imposed over the spheres models. In this case, we will say that a sentence is valid under a combination of conditions iff it is valid under every spheres interpretation that satisfies these conditions. Before introducing specific constraints on the models we will define “local” and “global” notions of consistency.

**Definition 1.4** A sentence \( A \) is consistent in a model \( M = \langle I, \$ \rangle \) and with respect to a valuation \( P \) for \( M \) iff \( P(A) \) is nonempty.

**Definition 1.5** \( A \) is consistent iff there is a model \( M \) such that \( A \) is consistent in \( M \).

The following is a list of constraints that will be used later in the proof:

(L) **Limit Assumption** \( \forall A \in L, \forall i \in I, \) if the proposition corresponding to \( A \) cuts \( \bigcup \$_i \), then there is some smallest member of \( \bigcup \$_i \) that overlaps the proposition \( A \).

(WC) **Weak Centering** \( \$ \) is weakly centered iff, \( \forall i \in I, i \) belongs to every nonempty member of \( \$_i \), and there is at least one nonempty member of \( \$_i \).

(SU) **Super Universality** If \( A \) is consistent, then for every model \( \langle I, \$ \rangle \) and for every \( i \in I \), \( P(A) \cap \bigcup \$_i \neq \emptyset \)

(UT) **Universality** \( \forall i \in I, \bigcup \$_i = I \)

The system \( \mathcal{V} \) presented in section 3 is complete and sound with respect to the class of system of spheres models constrained by (L). The system \( \mathcal{V} \) presented by Lewis.
is sound and complete with respect to the same class of models constrained by \((WC)\) and \((L)\). Finally the non-flat counterpart of the system \(\mathcal{EF}\) can be characterized by constraining the class of \(\mathcal{VW}\)-system of spheres models with \((SU)\).

We have to prove:

If \( ((A_1 > B_1), \ldots, \neg (A_i > B_i), \ldots, (A_n > B_n)) \models_{\mathcal{VW}} (A > B), \) then \( ((A_1 > B_1), \ldots, \neg (A_i > B_i), \ldots, (A_n > B_n)) \models_{\mathcal{V}} (A > B) \)

Assume the antecedent and suppose by contradiction that there is some \(\mathcal{V}\)-spheres model \(\langle I, \mathcal{S} \rangle\), such that for some \(i \in I\) there is a \(\mathcal{S}_i\) such that \(i \in \mathcal{P}(A_1 > B_1), \ldots, i \notin \mathcal{P}(A_i > B_i), \ldots, i \in \mathcal{P}(A_n > B_n)\), and \(i \notin \mathcal{P}(A > B)\). Since \(i \notin \mathcal{P}(A > B)\), we know that \(|\mathcal{S}_i|\) is nonempty. In addition, since \((L)\) holds we know that \(\mathcal{S}_i\) contains an innermost sphere (this innermost sphere will coincide with the smallest True-permitting sphere in \(\mathcal{S}(i)\)). Call this innermost sphere \(N\). Take any \(w \in N\) and consider the \(\mathcal{VW}\)-model \(\langle I, \mathcal{S}' \rangle\), such that \(\mathcal{S}'_w = \mathcal{S}_i\). The valuation \(\mathcal{P}'\) in \(\langle I, \mathcal{S}' \rangle\) is such that \(w \in \mathcal{P}(A_1 > B_1), \ldots, w \notin \mathcal{P}(A_i > B_i), \ldots, w \in \mathcal{P}(A_n > B_n)\), and \(w \notin \mathcal{P}(A > B)\). But this is impossible because we assumed \( ((A_1 > B_1), \ldots, \neg (A_i > B_i), \ldots, (A_n > B_n)) \models_{\mathcal{VW}} (A > B) \). Contradiction.

The proof of the coincidence of \(\text{GHF}(\mathcal{EF})\) and \(\text{GHF}(\mathcal{VW})\) is similar to the above proof.

\section*{D Main Proofs. Section 6}

\textbf{Observation 6.2} The set of \(\text{GHF}\) positive patterns generated by the epistemic model of \(\mathcal{V}\) coincides with \(\text{GHF}(\mathcal{V})\).

\begin{proof}
Assume that \(\sigma_1(A_1 > B_1), \ldots, \sigma_n(A_n > B_n) \models^{+}_{\mathcal{V}} (A > B)\), but \(\sigma_1(A_1 > B_1), \ldots, \sigma_n(A_n > B_n) \not\models^{+}_{\mathcal{V}} (A > B)\). Then, by Observation 3.10, \(\sigma_1(A_1 > B_1), \ldots, \sigma_{n-1}(A_{n-1} > B_{n-1}) \models^{+}_{\mathcal{V}} \sigma_n(A_n > B_n) \vdash (A > B)\). The process can be repeated \(n-1\) times more until the left-hand-side of \(\models^{+}_{\mathcal{V}}\) is empty, again by Obs. 3.10. Then we will have \(\models_{\mathcal{V}} \alpha\), where \(\alpha\) is the formula obtained after the repeated application of Observation 3.10.

Since \(\alpha\) is flat it must be a theorem in \(\text{Flat}(\mathcal{V})\) - by theorem 3.1 and Observations 3.1-5. But then by applying the deduction theorem \(n\) times in the opposite direction we get: \(\sigma_1(A_1 > B_1), \ldots, \sigma_n(A_n > B_n) \vdash_{\mathcal{V}} (A > B)\). The proof of the converse is similar.\end{proof}

\textsuperscript{**40}The other direction is trivial
E Main Proofs. Section 8

Theorem 8.1 $A \vdash_+ B$ iff $A \vdash_+ A > B$.

Proof: Assume $A \vdash_+ B$. Then every $K$ that supports $A$, also supports $B$. Therefore, for every $K$, the closest $A$-sets to $K$ will support $B$ (because all these sets support $B$), making $A > B$ positively valid.

Assume $A \vdash_+ A > B$. Take any $K$, such that $A$ is supported in $K$. Then, by (C), the minimal transformation of $K$ needed to make $K$ support $A$ is $K$ itself. But since $B$ is supported in the minimal transformation of $K$ needed to make $K$ support $A$, $B$ is supported by $K$.

Observation 8.1 When $B \in L_>$, $A \in L_0$, and EM is constrained by (K*3), ($A > B$) $\rightarrow$ ($A \rightarrow B$) is not $e_+$-valid.

Proof: Suppose by the sake of contradiction that ($A > B$) $\rightarrow$ ($A \rightarrow B$), is $e_+$-valid. Now notice that the following form of modus ponens holds: if $A \vdash_+ A$ and $A \vdash_+ A \rightarrow B$, then $A \vdash_+ B$. Therefore we can derive the following conditional:

If $A \vdash_+ B$, then $A \vdash_+ A \rightarrow B$.

For assume $A \vdash_+ B$. By theorem 8.1 $A \vdash_+ A > B$. Then by the assumed validity of (MP) (and the fact that $A \vdash_+ A \rightarrow B$ is entailed by $A \vdash_+ (A > B) \rightarrow (A \rightarrow B)$ and $A \vdash_+ A \rightarrow B$).

Now substitute $A \land B$ for $A$, and $A > B$ for $B$. We know (see section 3) that $A \land B \vdash_+ A > B$. But by Observation 3.8 $A \land B \rightarrow A > B$ is not positively valid.

Observation 8.2 (Pres) and (MPi) and (M) entail (AS).

Proof: By (MP) $\vdash (T > \neg A) \rightarrow (T \rightarrow \neg A)$. Then by classical propositional logic: $A \land B \vdash \neg(T > \neg A)$. Also by classical logic, $A \land B \vdash A \rightarrow B$. Therefore $A \land B \vdash (A \rightarrow B) \land \neg(T > \neg A)$. Now by (Pres) $A \land B \vdash ((A \rightarrow B) \land \neg(T > \neg A)) > (A > B)$. Finally by (MPi) and (M) $A \land B \vdash ((A \rightarrow B) \land \neg(T > \neg A)) \rightarrow (A > B)$. Therefore $A \land B \vdash (A > B)$.