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First Order Extensions of Classical Systems of Modal Logic

The role of the Barcan schemas

Abstract. The paper studies first order extensions of *classical* systems of modal logic (see (Chellas, 1980, part III)). We focus on the role of the *Barcan formulas*. It is shown that these formulas correspond to fundamental properties of neighborhood frames. The results have interesting applications in *epistemic logic*. In particular we suggest that the proposed models can be used in order to study monadic operators of *probability* (Kyburg, 1990) and *likelihood* (Halpern-Rabin, 1987).

Keywords: First order modal logic, Epistemic logic, Barcan formulas, Logic of likelihoood.

1. Introduction

The field known today as *epistemic logic* is almost fifty years old. The philosopher Georg von Wright proposed the study of the logics of knowledge and belief in a celebrated essay published in 1951 (von Wright, 1951). Eleven years later Jaakko Hintikka wrote an important book in the field (Hintikka, 1962). Hintikka's essay remains one of the best sources for understanding the nature of the enterprise in spite of the considerable amount of work done in this area during the last four decades. Hintikka was keenly aware of the special foundational problems that affect this branch of modal logic and devised formal solutions sensitive to these problems.

During the 60's and early 70's there was an explosion of work in epistemic logic. This work is summarized in (Lenzen, 1978). Many systems were studied both logically and conceptually and the adequacy of several special axioms scrutinized. During the last two decades of the past century a considerable amount of work in epistemic logic has been done in computer science and related areas. In fact, epistemic logics looked as ideal 'domain independent' tools for representing the knowledge of artificial or human agents. Many of the ideas proposed by Hintikka (and also by Dana Scott in his seminal 'Advice in modal logic' (Scott, 1970)) have been studied carefully. Hintikka and Scott insisted on the need to study both multi-modal

Presented by **Melvin Fitting**; *Final* April 29, 2001

Studia Logica **71**: 1–32, 2002.

and multi-agent logics. Single agents can be certain that something is the case or merely believe that, or they can claim that they know a proposition, or only judge it highly likely. In addition an agent can *claim* to have an attitude towards a proposition or it can *attribute* the attitude to a different agent. Finally a group of agents can commonly know something, etc. Multi-modal logics of belief, knowledge and certainty have been recently studied in detail and this study revealed interesting things about each attitude in isolation. By the same token, we have today a fairly firm grasp of multi-agent logics of knowledge and belief, and these formalisms have found interesting applications in distributed Artificial Intelligence (Fagin et al., 1995) and the Logic of Games (Parikh, 1998), (Pauli, 1998). This fact is, without doubt, a sign of progress that should be welcome. Nevertheless, many researchers agree that the most basic foundational problems that affected the logics of belief and knowledge since their inception are still with us. The following section will be devoted to providing a list of some of these problems. Then we will begin to articulate a solution extending the ideas first proposed by Montague and Scott in (Montague, 1970) and (Scott, 1970). Although this solution is well-known[1] we will suggest that it offers more than it is commonly assumed.

1.1. Some foundational problems in epistemic logic

Most modal semantics start with a space of 'possible worlds' or possible states of affairs. Informally the idea is to have a stock of primitives standing for all the ways the world might be. The metaphysics sustaining this idea can get rather subtle. Here we will only focus on the epistemological problems it poses.

In many applications even when the worlds in the model are posited as logical unstructured primitives, they are informally understood as composite entities. For example, it seems quite intuitive to consider worlds as structured $n{+}1$-tuples containing the nature state and n other coordinates specifying the epistemic states of the n agents. The nature state can have different encodings, depending on the considered application. In game theory it could be a strategy, in multi-agent systems an environment state of some form, etc.[2] What about the epistemic states of the agents? In fields like Economics knowledge is represented by a partition of worlds. Alter-

[1] At least this is so for propositional modal logics. See Part III of (Chellas, 1980) for a thorough presentation of this topic.

[2] See (Fagin et al., 1995, chapter 4) on *runs* and *systems* for applications in multi-agent systems. An informal explanation can be found in section 4.1.

natively the epistemic state of each agent can be represented by the set of propositions that the agent believes. In turn, each of these propositions is usually constructed as a set of worlds.

It is quite clear that the idea of interpreting worlds as composite entities (i.e. containing epistemic contexts as components) cannot be taken at face value on pain of circularity. In fact, the worlds used in the model would have worlds as components.[3] Jon Barwise (Barwise, 1988), Robert Aumann (Aumann, 1994) have recently suggested (independently) that the circularity can be seen as non-vicious by adopting an underlying set theory without the axiom of foundation.[4] This use of non-well founded set theory has been recently explored in detail by several researchers.

In epistemic logic there is a genuine need for a representation capable of keeping track of the epistemic state of the agent(s) that inhabit this or that world. But it seems that as long as such an epistemic state is encoded propositionally this type of representation cannot be taken at face value. Or it can be taken at face value only by giving up some important axioms of set theory.

Worlds can be construed as composite entities encoding the *local* states of agents together with the *environment* state as long as neither representation is propositional. For example, in multi-agent systems the local states might be encoded as sets of actions taken by agents at some point in time, or messages passed from agent to agent, or pairs of truth assignments and sets of formulas, etc.[5] But, local states cannot be encoded as propositions (or sets of propositions) without circularity, as long as we construe (as usual) propositions as sets of worlds themselves.

There is yet another alternative. The idea is to associate a *neighborhood* of propositions to each world. The intuition is that the set of propositions in the neighborhood of world Γ is the set of propositions accepted or believed by certain agent at Γ. Now the propositions in question are no longer *part* of Γ, they are *associated* with Γ. This is enough to break the circularity. At the same time the solution is expressive enough to keep track of the epistemic states of agents at worlds. This is the solution that we will explore in this essay. Of course, it is not a new solution to this and related problems in modal logic. Nevertheless, we will try to persuade the reader of the fruitfulness of this old idea. But before plunging into these arguments we should try to make explicit other problems of rival approaches.

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[3] Or sets of sets of such worlds, i.e. sets of propositions.

[4] See the chapter on Modal Logic in (Barwise-Moss, 1988) for a detailed account of this issue.

[5] See (Fagin et al., 1995, section 4.4) for several examples.

The dominant tool in modal semantics is the so-called Kripke semantics. Roughly the idea of this semantics is to add to a set of worlds (understood as primitives or points) an *accessibility relation.* A *frame* F is a pair $\langle W, R\rangle$ where W is a set of worlds and R an accesibility relation over W. A Kripke model $\mathcal{M}$ is a triple $\langle\langle W, R\rangle, \models\rangle$ where $\langle W, R\rangle$ is frame and $\models$ is a *valuation*, i.e. a relation between worlds and propositional letters. The valuation is extended beyond the propositional letter level in a standard manner for Boolean connectives. Then a proposition P is declared known at a certain point $\Gamma \in W$ in a model $\mathcal{M} = \langle\langle W, R\rangle, \models\rangle$ as long as P comes out true in all points accessible from Γ — in the frame $\langle W, R\rangle$. Relevant properties of the resulting knowledge operator will then be correlated with relevant properties of the accessibility relation. For example, the idea that knowledge entails truth would correspond to reflexivity, etc.

But according to this semantic account all knowledge operators should have certain properties and some of these properties have been found objectionable. We will focus here on the properties that correspond to what has been commonly called *logical omniscience* — the problem was first observed (and named) by Hintikka in (Hintikka, 1962). First notice that since a tautology holds true in every world we have that any tautology is known at every world. Secondly if P is known and P entails Q, then Q should be known as well. Finally if P is known and Q is known, then their conjunction P $\wedge$ Q is also known. The latter property is usually called *Adjunction.*

Many researchers think that those properties of knowledge are too strong. The reasons for this vary but before entering this let's first point out rapidly that none of them needs to be postulated as essential in neighborhood semantics.

A frame in neighborhood semantics is a pair $\langle W, N\rangle$ where W is a set of worlds and N a function from elements of W to sets of sets of worlds. When we have a world Γ and an associated neighborhood $N(\Gamma)$ there is a very simple manner of determining if a proposition P $\subseteq W$ is known at Γ in the corresponding model. We just check whether P is in $N(\Gamma)$. If the answer is yes, then P is indeed known. Otherwise it is not known. But then it is rather simple to see that none of the forms of logical omniscience previously considered is now obligatory.[6] Nothing requires that the set of worlds W is part of $N(\Gamma)$. On the other hand if a proposition P is in a neighborhood, and Q is a superset of P, Q need not be in the neighbor-

[6] A weak form of logical omniscience, not considered above, continues to hold. When A and B are logically equivalent, A should be known if and only if B is known. The status of this form of logical omniscience will be discussed below in section 5.

hood. Finally even if both P and Q are in $N(\Gamma)$P $\cap$ Q need not be in the neighborhood.

It is a rather contentious issue from the philosophical point of view whether a knowledge operator should or should not obey the properties that determine logical omniscience. With some provisos some authors think that none of these properties should be required, others think that all of them should be imposed and, finally, some think that only a subset of these properties are required. More importantly these judgements vary considerably depending on the studied attitude (knowledge, belief, likelihood, etc) and depending on whether the formalized concept intends to capture either *epistemic attributions* made about a third agent or *epistemic claims* made by the agent himself.

It is perhaps less controversial that *some* of the properties in question are problematic for *some* epistemic notions, under *all* interpretations of the operator. For example, Adjunction is an inadequate property for a monadic epistemic operator capturing the property of 'high probability' — or 'the probability of P is greater than a certain threshold t'. It is clear that even when the probability of propositions P and Q surpasses the threshold t, their conjunction might bear a probability below t. We will argue below that only a minor modification of the standard presentation of neighborhood semantics suffices to provide an adequate tool to encode a family of monadic notions of likelihood.[7]

[7] Strictly speaking the likelihood of a hypothesis H on data e (L(H/e)) is P(e/H), where P is a standard probability function. But, of course, a qualitative notion of probability or likelihood can also be studied. Our model diverges, however, from accounts like Savage's (Savage, 1972), who focuses on clarifying the two-place operator 'the event p is more likely than q'. Logics of these type have been recently studied by modal logicians ((Gardenfors, 1975) presents an interesting analysis of the notion of probability as an intensional operator, as well as references to previous work in this area). Nevertheless, in these formalisms there is no direct way of saying 'p is likely'. In this paper we will consider first order extensions of a family of classical modal logics that, we argue, articulate minimal axioms shared by a large family of qualitative notions of likelihood. 'H is probably to a sufficient degree' is one of these notions. But, as (Halpern-Rabin, 1987) suggest, 'probability theory is not the only way to reason about likelihood' (see (Halpern-Rabin, 1987, p. 381)). 'H is likely' can also be qualitatively understood as 'H is possible to a sufficient degree' (Dubois-Prade, 1992); or as 'H is expected to a sufficient degree' (Gardenfors-Makinson, 1994). The former notion corresponds to saying 'H is unsurprising to a degree sufficient to escape rejection' (Shackle, 1952), and this, in turn, corresponds to saying that 'H is not expected to a sufficient degree'. There are, to be sure, important differences separating these qualitative notions of likelihood. Moreover one can argue that the notion 'H is probable to a sufficient degree' radically differs from the other notions, just in virtue of the fact of being non-Adjunctive — while one might argue that the other notions of likelihood can be represented as obeying Adjunction. Our point here is that whatever the

So, the model that we are about to present has two main features. On the one hand it facilitates the study of useful epistemic models where Logical Omniscience fails. On the other hand it offers a solution to the problem we mentioned at the beginning of this section, namely the need of keeping track of epistemic states of agents in possible worlds.[8] Summing up: we will argue that the proposed model makes possible the study of an interesting family of epistemic notions.

An important part of this article will be devoted to arguing that the use of neighborhood semantics makes possible the study of the interesting role of the so-called Barcan formulas in epistemic logic. The use of neighborhood semantics is particularly useful for studying the role of those formulas in logics of certainty and likelihood. We will also argue that this study is beyond the expressive power of Kripke models. Establishing the result will require to extend Montague-Scott's models to the first-order case. We will do this in the following sections after providing some background.

From a logical point of view we will show that the Barcan formulas correspond to fundamental properties of neighborhood frames. We will also show that these properties are quite different from the equally fundamental, but differently motivated, properties that the Barcan formulas induce in Kripke frames. From an applied point of view we will argue that such properties reveal interesting aspects of epistemic logics where logical omniscience fails.

2. Propositional modal logic

We will introduce here the basis of the so-called *neighborhood semantics* for propositional modal logics. We will follow the standard presentation given in Part III of (Chellas, 1980).

A warning about nomenclature is needed at this juncture. Chellas' name for Montague-Scott structures is *minimal models* rather than neighborhood models. In order to keep the terminology simple we will continue to refer to Montague-Scott's structures as neighborhood models.

DEFINITION 2.1. $\mathcal{M} = \langle W, N, \models \rangle$ is a neighborhood model if and only if:

(1) W is a set,

(2) N is a mapping from W to sets of subsets of W,

(3) $\models$ is a relation between possible worlds and propositional letters.

structural differences separating the different manners of understanding likelihood, those differences can be naturally captured in the framework we are about to present.

[8] To be more precise, it allows to keep track of propositional representations of epistemic states of agents in possible worlds.

Of course the pair $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$ is a *neighborhood frame*. The notation $\mathcal{M}, \Gamma \models A$ is used to state that A is true in the model $\mathcal{M}$ at world Γ. The following definition makes precise the notion of truth in a model.

DEFINITION 2.2 (Truth in a neighborhood model). Let Γ be a world in a model $\mathcal{M} = \langle W, N, \models \rangle$. $\models$ is extended to arbitrary formulas in the standard way for Boolean connectives. Then the following clauses are added in order to determine truth conditions for modal operators.

(1) $\mathcal{M},\ \Gamma \models \Box A$ if and only if $|A|^{\mathcal{M}} \in N(\Gamma)$,

(2) $\mathcal{M}, \Gamma \models \Diamond A$ if and only if $|\neg A|^{\mathcal{M}} \notin N(\Gamma)$.

where, $|A|^{\mathcal{M}} = \{\Gamma \in W : \mathcal{M}, \Gamma \models A\}$.

$|A|^{\mathcal{M}}$ is called A's *truth set*. Intuitively $N(\Gamma)$ yields the propositions that are necessary at Γ. Then $\Box A$ is true at Γ if and only if the 'truth set' of A (i.e. the set of all worlds where A is true) is in $N(\Gamma)$. If the intended interpretation is epistemic $N(\Gamma)$ contains a set of propositions understood as epistemically necessary. This can be made more precise by determining the exact nature of the epistemic attitude we are considering. $N(\Gamma)$ can contain the known propositions, or the believed propositions, or the propositions that are considered highly likely, etc. Then the set $\mathrm{P} = \{A \in W : \models \Diamond A\}$ determines the space of epistemic possibilities with respect to the chosen modality — knowledge, likelihood, etc.

Clause (2) forces the duality of possibility with respect to necessity. It just says that $\Diamond A$ is true at Γ if the denial of the proposition expressed by A (i.e. the complement of A's true set) is not necessary at Γ. As we said above, $N(\Gamma)$ is called the *neighborhood* of Γ.

3. Augmentation

The following conditions on the function N in a neighborhood model $\mathcal{M} = \langle W, N, \models \rangle$ are of interest. For every world Γ in $\mathcal{M}$ and every proposition (set of worlds) X, Y in $\mathcal{M}$:

(m) if $X \cap Y \in N(\Gamma)$, then $X \in N(\Gamma)$, and $Y \in N(\Gamma)$;

(c) if $X \in N(\Gamma)$, and $Y \in N(\Gamma)$, then $X \cap Y \in N(\Gamma)$.

(n) $W \in N(\Gamma)$.

When the function N in a neighborhood model satisfies conditions (m), (c) or (n), we say that the model is *supplemented*, is *closed under intersections*, or *contains the unit* respectively. If a model satisfies (m) and (c) we say that is a *quasi-filter*. If all three conditions are met it is a *filter*. Notice that filters can also be characterized as non-empty quasi-filters — non-empty in the sense that for all worlds Γ in the model $N(\Gamma) \neq \emptyset$.

DEFINITION 3.1. A neighborhood model $\mathcal{M} = \langle W, N, \models \rangle$ is *augmented* if and only if it is supplemented and, for every world Γ in it:

$$\bigcap N(\Gamma) \in N(\Gamma).$$

Now we can present an observation (established in (Chellas, 1980), section 7.4), which will be of heuristic interest in the coming section.

OBSERVATION 3.1. *$\mathcal{M}$ is augmented just in case for every world Γ and set of worlds X in the model:*

$$X \in N(\Gamma) \quad \textit{if and only if} \quad \bigcap N(\Gamma) \subseteq X \tag{BS}$$

3.1. Epistemic interpretation of augmentation

In recent work in epistemic logic it is quite usual to represent agents by *acceptance sets* or *belief sets*, obeying certain rationality constraints. If the representation is linguistic the agent is represented by a logically closed set of sentences. If the representation is done in a sigma-field or relative to a universe of possible worlds, the agent is represented by a set of points such that all propositions accepted (believed) by the agent are supersets of this set of points. Adopting either representation is tantamount to imposing logical omniscience as a rationality constraint.

When a neighborhood frame is augmented we have the guarantee that, for every world Γ, its neighborhood $N(\Gamma)$ contains a smallest proposition, composed by the worlds that are members of every proposition in $N(\Gamma)$. In other words, for every Γ we know that $N(\Gamma)$ always contains $\bigcap N(\Gamma)$ and every superset thereof (including W).

We will propose to see the intersection of the neighborhood of a world as an acceptance set for that world, obeying the rationality constraints required by logical omniscience. The following results help to make this idea more clear.

OBSERVATION 3.2. *If $\mathcal{M}$ is augmented, then for every Γ in the model:*

(1) $\Gamma \models \Box A$ *iff and only if* $N(\Gamma) \subseteq |A|^{\mathcal{M}}$,

(2) $\Gamma \models \neg\Box A$ *iff and only if* $N(\Gamma) \not\subseteq |A|^{\mathcal{M}}$.

PROOF. Assume $\Gamma \models \Box A$. Then we have $|A|^{\mathcal{M}} \in N(\Gamma)$. But if the model is augmented: (BS).

So, by (BS), we have $\bigcap N(\Gamma) \subseteq |A|^{\mathcal{M}}$ as desired. The converse also follows immediately from (BS). (2) also follows immediately from (BS). ■

Epistemic possibility is, in this setting, understood in terms of compatibility with the belief set $\bigcap N(\Gamma)$. In other words $\models_{\mathcal{M}} \Diamond A$ if and only if $|A|^{\mathcal{M}} \bigcap (\bigcap N(\Gamma)) \neq \emptyset$. This in turn means that, when the model $\mathcal{M}$ is augmented, $\models_{\mathcal{M}} \Diamond A$ holds whenever $|A|^{\mathcal{M}}$ is logically compatible with *every* epistemically necessary proposition in the neighborhood.

4. Epistemic possibility and likelihood

When the notion of epistemic necessity is understood as 'highly likely' (or 'probable enough') many authors have recommended not using augmented models in order to represent those judgements. Here is an example, adapted from a similar one presented by Henry Kyburg in (Kyburg, 1990), that can help us illustrate the motivations of such authors.

Suppose your job is measuring items produced by a certain machine. They are OK or they are not. Of course measurements cannot be error-free. But it can be demanded that false assessments of OK should bear probability less than (say) .0001. You should therefore consider highly likely, of each inspected piece, that it is OK.

It is also clear that you ought to consider highly likely that of a large number of items at least one will not be OK. Thus if we represent the judgements of likelihood in terms of membership to a neighborhood N we are in a situation in which if the neighborhood were augmented it would be inconsistent (in the sense in which the falsity would be judged likely). Proposed solutions to the problem are, in this framework tantamount to giving up condition (c) — this is, in essence, the solution proposed in (Kyburg, 1961).

What is the notion of epistemic possibility that is appropriate in this situation? We propose to apply here the same criterion which clearly applies when the model is augmented, namely that A is possible (seriously possible) as long as it overlaps *every* judgement of likelihood in the neighborhood. In other words, the proposition $|A|^{\mathcal{M}}$ expressed by A is seriously impossible at Γ as long as it fails to overlap *some* member X of the neighborhood $N(\Gamma)$.

This seems to be Kyburg's own idea about this matter. Say that K is the set of sentences receiving high probability. For each world Γ, the set K would be construed in our setting as the set of sentences such that $\Box A$ holds true at some point Γ. Then, commenting on the previous example Kyburg points out in (Kyburg, 1990):

> The deductive closure of K contains all the statements of the language. Yet I have no difficulty using this (unclosed!) set of statements K for planning, or as a standard for serious possibility. Even though the *conjunction* of statements in K is impossible, K can serve as a standard of serious possibility in the sense that if a statement contradicts a *member* of K, it is not to be regarded as a serious possibility. That one of the pieces has passed my inspection is not OK is not a serious possibility. And at the same time, it is not a serious possibility that *all* the inspected pieces are OK.

Notice that if we officially define for a model $\mathcal{M}$:

DEFINITION 4.1. (Poss) $\mathcal{M}, \Gamma \models \Diamond A$ if and only if for every X in $N(\Gamma)$, $|A|^{\mathcal{M}} \cap X \neq \emptyset$.

Then *if the model in question is augmented* the previous definition collapses with the standard definition provided above (2.2.2). This is easy to see. We can sketch one half of the proof to make things transparent. Say that the underlying model is augmented and that $|A|^{\mathcal{M}}$ overlaps all members of the neighborhood $N(\Gamma)$. Then it also overlaps the intersection of the neighborhood. As a result $|\neg A|^{\mathcal{M}}$ cannot be a superset of $\bigcap N(\Gamma)$. So, $|\neg A|^{\mathcal{M}}$ is not in $N(\Gamma)$ and this entails that A is possible in the standard sense.

Nevertheless the previous definition of possibility does not collapse with the standard one when the model fails to be augmented. A variant of our example on measurement will suffice to make this clear. Say that the universe of worlds encompasses all possible combinations of failures and OK cases. We have an 'extreme' world where all pieces are faulty. If there are n pieces call this world *fn*. The world where all the pieces are OK can be called OKn. Assume in addition that the neighborhood of the actual world contains all propositions of the form 'piece i is OK' for i between 1 and n. This reflects the fact that it is highly likely that each piece is OK. We can have as well the proposition 'at least a piece is not OK' in the neighborhood.

We might also have the proposition 'at least a piece is OK' in the neighborhood. But for some subsets S of the total set of n pieces the neighborhood might lack propositions of the form 'at least a piece in S is OK'. This is one of the prices that has to be paid for the lack of logical omniscience.

Take now one of these subsets S of the total set of n pieces and consider the proposition 'all pieces in S are faulty' $= P$. One does not want to consider such proposition as seriously possible — when S is large. Nevertheless the standard definition of possibility will make P seriously possible independently of how large S is — just because the complement of P does not happen to be in the neighborhood. Our definition improves on the standard one in this and other cases. Notice that P will be seriously impossible according to (Poss). In fact it will not be compatible with any of the propositions 'piece i is OK' for pieces i in S — such propositions are constructed as sets of worlds all of which make true the sentence stating that i is OK.

The sentence stating that all pieces in S are faulty is not possible for Kyburg either. In fact, this sentence is inconsistent with several members of the set K used by Kyburg. Just take any sentence in K stating that a piece in S is OK.

The conclusion is that the standard definition of possibility in classical neighborhood semantics seems inadequate. In fact, the definition of possibility and necessity should work well in the cases where logical omniscience fails. It turns out that the standard definition is adequate exactly when logical omniscience *does not* fail. We therefore propose to adopt our own Definition 4.1 instead. Of course the standard definition has been proposed without considering the particular epistemic interpretation we have been focusing on. But, taking into account the centrality that the epistemic interpretation seems to have in this case, it seems advisable to reform the definition of possibility in the general case.

5. Classical systems of modal logic, logical omniscience and duality

Neighborhood semantics characterizes a family of so-called *classic* systems of modal logic (see PART III of (Chellas, 1980)).

Definition 5.1. A system of modal logic is *classical* if and only if it contains the axiom $\Diamond A \leftrightarrow \neg\Box\neg A$, and is closed under the rule of inference **RE**, according to which $\Box A \leftrightarrow \Box B$ should be inferred from $A \leftrightarrow B$.

Neighborhood semantics does sanction a weak principle of logical omniscience according to which any agent who knows (believes, etc) A should also know (believe, etc) B, as long as A and B are logically equivalent. The syntactical clothing adopted by the proposition expressed by either A or B is irrelevant.[9]

[9] If the principle is adopted to represent *commitments to knowledge* (belief) of rational agents, then it seems adequate. But one might argue that all forms of logical omniscience

It is unclear how much of logical omniscience should be abandoned when part of it has already been abandoned. A careful analysis of these issues is beyond the scope of this piece. Nevertheless it should be said that closure under logical equivalence is not a form of logical omniscience questioned by researchers interested in studying logics of likelihood. The main target is Adjunction. So, for our main purposes here neighborhood semantics will do, even when it does not manage to deactivate all forms of logical omniscience.

Seven systems of classical modal logic are weaker than the weakest system characterizable in Kripkean semantics (the system **K**). The weakest of those systems is **E** — we follow here the terminology presented in (Chellas, 1980). Then three classical systems are obtained by adding supplementation, closure under (finite) intersection, and the unit principle to the basic semantics. When supplementation is added we have the system **EM** by adding the following axiom to **E**:

$$\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B) \tag{M}$$

When closure under finite intersections is added we have the system **EC** validating:

$$(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B) \tag{C}$$

If the neighborhood contains the unit we have **EN** by adding the following axiom to E:

$$\Box \mathsf{True} \tag{N}$$

Finally we also have the systems **EMC**, **EMN** and **ECN**, all of which are weaker than the weakest Kripkean system, which, in this hierarchy is **EMCN** — again, we use the nomenclature in chapter 8 of (Chellas, 1980).

5.1. Duality

Of course, the adoption of the notion of possibility proposed in Definition 4.1 will make a difference when it comes to characterize classical systems of modal logic. The standard definition of possibility establishes the duality of possibility and necessity by fiat. Our definition only preserves half of the bi-conditional establishing this duality at the syntactic level. In fact we have

$\leftrightarrow$? the axiom:

$$\Diamond A \rightarrow \neg\Box\neg A \tag{DEF-Poss}$$

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are adequate for this type of normative modeling. The status of the principle is more precarious for syntactical representations of the epistemic performance of bounded agents.

So, our semantics characterizes the smallest system that contains (DEF-Poss) and that is closed under **RE**. We will call this system **EP**. Such system is clearly weaker than the weakest system of classical modal logic **E**.

In the previous sections we presented a principled argument in favor of adopting a modified definition of possibility in neighborhood semantics, especially for epistemic applications. We can rewrite our definition as follows:

DEFINITION 5.2 (DEF-Poss). $\mathcal{M}, \Gamma \models \Diamond A$ if and only if $\forall X \in N(\Gamma)$, $\exists \Delta \in X$, $\mathcal{M}, \Delta \models A$.

Nothing has been said about the intuitiveness of the semantic characterization of necessity, which we left unchanged. For applications where it makes sense to preserve the duality of necessity and possibility, and where our characterization of possibility is reasonable, the semantic characterization of necessity can be changed accordingly:

DEFINITION 5.3 (DEF-Nec). $\mathcal{M}, \Gamma \models \Box A$ if and only if $\exists X \in N(\Gamma)$, $\forall \Delta \in X$, $\mathcal{M}, \Delta \models A$.

The definition of necessity given in Definition 5.3 has been proposed in passing in (Chellas, 1980, Exercise 7.9, p. 211) as an alternative to the classical truth conditions for *minimal* models presented above (Definition 2.2). Chellas does not provide, however, further intuitive support for these alternative truth conditions. In the light of the above arguments one might consider that they are required in epistemic applications, provided that the duality of necessity and possibility is also motivated. In this article we will focus on first order extensions of the standard minimal operators (characterized semantically in Definition 2.2). We will also consider some effects of adopting the classical definition of necessity and DEF-Poss in the first order setting. It would be also interesting to study the first order extension of the operators characterized via DEF-Poss and DEF-Nec, but this issue will not be considered here.

6. Neighborhood models for first order modal logic

A considerable amount of work has recently been done studying first and higher order extensions of modal logic (see (Hughes-Cresswell, 1996) as well as (Fitting-Mendelsohn, 1999) for recent textbook presentations of this work). At the same time the classical definability of modal formulas (viewed as relational principles) has also been studied in detail (van Benthem, 1984). Nevertheless less work has been done studying the first order extensions of epistemic logics. Most of the developed extensions of modal logic focus on other interpretations of the modals (alethic, temporal, etc). In addition neighborhood semantics has not been extended in order to deal with the phenomenon of quantification. The recent work on first order modal logics focuses instead on the first order extensions of Kripkean systems of modal logic.

A notable exception is Hintikka's own essay, which devotes its last chapter to the problems posed by existence and epistemic modality. Semantically Hintikka does not appeal either to possible world semantics or to neighborhood semantics. He offers his own analysis, which can be seen as a precursor to a disproof procedure (anticipating recent work on tableaux systems). Hintikka does tackle some issues of special interest for epistemic applications, although his agenda of open problems in this field does not overlap with ours. In the area of computer science the interest in first order extensions of epistemic systems is also quite marginal (the entire reference to this issue in (Fagin et al., 1995) is reduced to section 3.7).

Our goal in this final part of the article is to present a first order extension of neighborhood semantics and to apply it to consider the epistemic role of the so-called Barcan schemas. We will argue that the extension provides interesting insights both about the epistemic uses of these schemas and about the nature of likelihood operators. Moreover it will allow us to formulate necessary and sufficient conditions for the representation of closed belief sets in neighborhoods.

6.1. Constant domain models and the Barcan schemas

We introduce first the basic definitional apparatus needed to extend neighborhood semantics to the first order case. We follow the basic terminology proposed by Fitting and Mendelsohn in (Fitting-Mendelsohn, 1999).

DEFINITION 6.1. A structure $\langle W, N, D\rangle$ is a *constant domain neighborhood* frame if $\langle W, N\rangle$ is a neighborhood frame and D is a non-empty set, called the *domain* of the frame.

In this case all the worlds in W have the same domain. The alternative would be to build models where the domains vary across worlds. The resulting *varying domain models* will be briefly considered below.

The following schemas have an interesting pedigree in first-order modal logic. They implement some basic forms of quantifier/modality permutability.

DEFINITION 6.2. All formulas of the following form are Barcan formulas:

$$(\forall x)\Box\phi \rightarrow \Box(\forall x)\phi$$
$$\Diamond(\exists x)\phi \rightarrow (\exists x)\Diamond\phi$$

The implications going the other way give us:

DEFINITION 6.3. All formulas of the following form are Converse Barcan formulas:

$$\Box(\forall x)\phi \rightarrow (\forall x)\Box\phi$$
$$(\exists x)\Diamond\phi \rightarrow \Diamond(\exists x)\phi$$

Why are the Barcan schemas important? When Ruth Barcan Marcus first studied them in (Barcan Marcus, 1946) modal semantics (in any of the forms previously mentioned here) had not yet been invented. So Marcus first studied them under a purely syntactic point of view. Marcus also pioneered a model-theoretical analysis of the formulas. In (Barcan Marcus, 1961) she showed the validity of $\Diamond(\exists x)\phi \rightarrow (\exists x)\Diamond\phi$ in first order *models* with constant domains of the type outlined in (Carnap, 1947).[10] In recent years we have learnt more about the role that these schemas play in Kripkean semantics. In fact it turns out that the Barcan and Converse Barcan schemas correspond to simple conditions on Kripke *frames* (see (Fitting-Mendelsohn, 1999, section 4.9)).

Call *locally constant* a varying domain Kripke frame such that if a pair of worlds are in the in the domain of the accessibility relation of the frame, then their domains coincide. It turns out that the validity of both the Barcan and Converse Barcan schemas corresponds exactly to the locally constant domain condition on Kripke frames. So, the Barcan schemas seem to correspond to fundamental semantic properties of Kripke frames. In fact, the

[10] The gist of the idea here is to consider the class of first order models of an underlying language L (equipped with constants). The class of models is defined with respect to a fixed domain of objects. The semantic construction outlined by Marcus in (Barcan Marcus, 1961) seems to determine the universal domain as the set of objects named by the constants of L in the actual situation. Then $\Diamond\phi$ is true at a model M if and only if ϕ is true in some model M_1. As Marcus points out, although this semantic construction appeals to the Leibnizian distinction between true in a possible world and true in all possible worlds, it is also true that the construction does not posit specifically intensional objects. 'No new entity is spawned in a possible world that isn't already in the domain of terms of which the class of models is defined' (see (Barcan Marcus, 1961, p. 23) in the reprinted version). For our own purposes it seems clear that Marcus' construction is sufficient to prove the validity of the Barcan schemas in a first order modal semantics with constant domain.

Barcan schemas (and Converse Barcan) go much beyond the issue of quantifier/modal permutation. They really make explicit important existence assumptions in Kripkean semantics. Fitting and Mendelsohn put this in a crisp way:

> The Converse Barcan formula says that, as we move to an alternative situation, nothing passes out of existence. The Barcan formula says that, under the same circumstances, nothing comes into existence. The two together say the same things exist no matter what the situation.
> (Fitting-Mendelsohn, 1999, p. 114)

What is the role of the Barcan and Converse Barcan formulas in Montague-Scott's semantics? Do they correspond to interesting conditions in neighborhood frames? We will tackle this question first. We will see that Barcan and Converse Barcan do correspond to interesting conditions in neighborhood frames. Although Barcan and Converse Barcan are really schemas we will follow standard terminology and we will call them formulas, unless an explicit clarification is needed.

6.2. The role of the Barcan formulas in neighborhood semantics

We need some introductory definitions first. Inessential details are skipped. We will only comment on novel definitions.

DEFINITION 6.4. A constant domain neighborhood frame $\langle W, N, D\rangle$ is *augmented* if $\langle W, N\rangle$ is augmented.

CDN will stand from now on for 'constant domain neighborhood'. Now we can move from frames to models by introducing the classical notion of interpretation:

DEFINITION 6.5. $\mathcal{I}$ is an interpretation in a CDN $\langle W, N, D\rangle$ if $\mathcal{I}$ assigns to each n-place relation symbol R and to each point (in W), some n-place relation on the domain D of the frame.

DEFINITION 6.6. A CDN model is a structure $\langle W, N, D, \mathcal{I}\rangle$, where the triple $\langle W, N, D\rangle$ is a CDN frame and $\mathcal{I}$ is an interpretation in it.

DEFINITION 6.7. Let $M = \langle W, N, D, \mathcal{I}\rangle$ be a CDN model. A valuation in the model M is a mapping v that assigns to each variable x some member $v(x)$ of D. w is a x-variant of a valuation v if w and v agree on all variables except possibly the variable x.

Now we can introduce the crucial notion of truth in neighborhood models. The last two conditions deal with (a classical characterization of) quantification. For the moment we will not modify the standard definition of possibility. After presenting two central results about the conditions imposed by Barcan schemas in neighborhood frames, we will consider the impact of reforming the characterization of epistemic possibility (via the adoption of Definition 4.1 (Poss)).

DEFINITION 6.8 (Truth in a model). Let $\mathcal{M} = \langle W, N, D, \mathcal{I} \rangle$ be a CDN model. For each Γ in W and valuation v in the model:

(1) If R is an n-place relation symbol, $\mathcal{M}, \Gamma \models_v R(x_1, ..., x_n)$ provided $\langle v(x_1), ..., v(x_n) \rangle \in \mathcal{I}(R, \Gamma)$.

(2) Standard valuations for negation and binary connectives.

(3) $\mathcal{M}, \Gamma \models_v \Box A$ if and only if $|A|^{\mathcal{M},v} \in N(\Gamma)$.

(4) $\mathcal{M}, \Gamma \models_v \Diamond A$ if and only if $|\neg A|^{\mathcal{M},v} \notin N(\Gamma)$.

(5) $\mathcal{M}, \Gamma \models_v (\forall x)\phi$ if and only if for every x-variant w of v in $\mathcal{M}$: $\mathcal{M}, \Gamma \models_w \phi$.

(6) $\mathcal{M}, \Gamma \models_v (\exists x)\phi$ if and only if for some x-variant w of v in $\mathcal{M}$: $\mathcal{M}, \Gamma \models_w \phi$.

We follow here the standard notations and conventions presented in (Fitting-Mendelsohn, 1999, section 4.6). We make explicit only some basic extensions of the notion of truth set used above in the propositional case.

$$|R(x_1, ..., x_n)|^{\mathcal{M},v} = \{\Gamma \in W : \mathcal{M}, \Gamma \models_v R(x_1, ..., x_n)\}.$$

More about terminology. Call *non-trivial* a frame whose domain contains more than one object.

OBSERVATION 6.1. *A non-trivial constant domain neighborhood frame is supplemented if and only if every model based on it is one in which the Converse Barcan formula is valid.*

PROOF. Assume $\mathcal{M}$, $\Gamma \models_w \Box(\forall x)F(x)$. Given the assumption, we have that $\bigcap\{|F(x)|^{\mathcal{M},w} : w \text{ is a } x\text{-variant of } v\} = |(\forall x)F(x)|^{\mathcal{M},v} \in N(\Gamma)$. So, for each x-variant w of v, supplementation guarantees that $|F(x)|^{\mathcal{M},w} \in N(\Gamma)$. Therefore, $\mathcal{M}, \Gamma \models_w (\forall x)\Box F(x)$. Since in order to establish this argument $F(x)$ could have been *any* formula, this proves the LTR part of the observation.

Suppose now that the frame $\langle W, N, D\rangle$ is not supplemented. Therefore we have truth sets X and Y such that $X \cap Y \in N(\Gamma)$, but either X is not in $N(\Gamma)$ or Y is not in $N(\Gamma)$. We also assume non-triviality of the domain, which guarantees that there are at least two distinct objects a and b in D. We will construct a family of models based on such frame where one particular Converse Barcan formula is not valid. The models in question satisfy the following constraints:

(1) For every world Δ in $X \cap Y$, and for all c in the domain of the frame, $\langle c\rangle \in \mathcal{I}(F, \Delta)$.

(2) For every world Γ in $X - (X \cap Y)$ there is c' in D such that $\langle c'\rangle \in \mathcal{I}(F, \Gamma)$, but there is no other c in D, such that $\langle c\rangle \in \mathcal{I}(F, \Gamma)$. For every world Γ in $Y - (X \cap Y)$ there is $c'' \neq c'$ in D such that $\langle c''\rangle \in \mathcal{I}(F, \Gamma)$, but there is no other c in D, such that $\langle c\rangle \in \mathcal{I}(F, \Gamma)$.

(3) $\mathcal{I}(F, \Gamma')$ is empty for worlds Γ' in $W - (X \cup Y)$.

$\bigcap\{|F(x)|^{\mathcal{M},v} : w \text{ is a } x\text{-variant of } v\} = |(\forall x)F(x)|^{\mathcal{M},v} \in N(\Gamma)$, but it is
changes not true that for all x-variants w of v we have:

$$\mathcal{M}, \Gamma \models_w \Box F(x)\,.$$

In fact, without loss of generality we can assume that $X \notin N(\Gamma)$. Then, for $w(x) = c'$, $|F(x)|^{\mathcal{M},w} = X \notin N(\Gamma)$. But, the above constraints guarantee that:

$$\mathcal{M}, \Gamma \models_w \Box(\forall x)F(x)$$

In fact, $X \cap Y = |(\forall x)F(x)|^{\mathcal{M},v} \in N(\Gamma)$. Non-triviality is used when we assumed that the underlying domain contains at least two distinct objects c'' and c'. ■

A frame is *consistent* if and only if for every Γ in W, $N(\Gamma) \neq \emptyset$ and $\{\emptyset\} \notin N(\Gamma)$. In other words a frame is consistent as long as the agent represented by the frame does not believe the falsity — and the neighborhoods of the frame are not empty. Call *regular* any frame that is consistent and non-trivial.

OBSERVATION 6.2. *A regular constant domain neighborhood frame is closed under finite intersections*[11] *if every model based on it is one in which the Barcan formula is valid.*

[11] See condition (c) in section 3 above.

PROOF. As in the previous observation we will assume that a regular frame is not closed under finite intersections. Then we will construct a family of models based on such frame where a particular Barcan formula is not valid. The models in question satisfy the following constraints:

Focus then on two arbitrarily chosen beliefs X and Y in $N(\Gamma)$, where $Y \neq X$ and $X \cap Y \notin N(\Gamma)$. Since we are assuming that the frame is not closed under finite intersections we know that these two beliefs (X and Y) exist. Then for every Δ in each $X - Y$ pick an object c^x in the domain and establish that $\langle c^x \rangle \in \mathcal{I}(F, \Delta)$, while for every other object c different from c^x, $\langle c \rangle \notin \mathcal{I}(F, \Delta)$. Establish in addition that for every Δ in Y, $\langle c^x \rangle \notin \mathcal{I}(F, \Delta)$, while for every other object c different from c^x, $\langle c \rangle \in \mathcal{I}(F, \Delta)$. So, if $X \cap Y$ is non-empty, every world in $X \cap Y$ is such that F is true of every object of the domain at that world. Finally let $\mathcal{I}(F, \Delta')$ be empty for any other world Δ' in W.

Regularity (non-triviality) guarantees that there are at least two distinct constants in the domain of the model, which is enough to carry out the correspondent part of the previous construction. Since we assumed that $X \cap Y$ is not in $N(\Gamma)$, $\Box(\forall x)F(x)$ fails to be true at Γ. Nevertheless, for every x-variant w of v, we have that:

$$\mathcal{M}, \Gamma \models_w \Box F(x)\,.$$

There are two cases we can consider. Either $w(x) = c^x$ or $w(x) = c$, with $c \neq c^x$. In the first case, $w(x) = c^x$, $|F(x)|^{\mathcal{M},w} = X \in N(\Gamma)$. In the second case, $w(x) = c$, $|F(x)|^{\mathcal{M},w} = Y \in N(\Gamma)$. If $X \cap Y$ is empty then also $\Box(\forall x)F(x)$ fails to be true at Γ, because $|(\forall x)F(x)|^{\mathcal{M},v} \notin N(\Gamma)$ — in virtue of the assumed consistency of the frame. ■

OBSERVATION 6.3. *If a finite constant domain neighborhood frame is closed under intersections then every model based on it is one in which the Barcan and Converse Barcan schemas are valid.*

PROOF. We will show that if $\langle W, N, D \rangle$ is a finite *constant domain neighborhood* frame then:

$$\text{If } \cap N(\Gamma) \in N(\Gamma) \text{ then } \mathcal{M}, \Gamma \models_w (\forall x)\Box F(x) \to \Box(\forall x)F(x).$$

Assume that $\mathcal{M}, \Gamma \models_w (\forall x)\Box F(x)$. Therefore for all x-variants w of v we have that $|F(x)|^{\mathcal{M},w} \in N(\Gamma)$. Since the neighborhood is assumed to be closed under intersections $\bigcap\{|F(x)|^{\mathcal{M},w}\colon\ w \text{ is a } x\text{-variant of } v\} = |(\forall x)F(x)|^{\mathcal{M},v} \in N(\Gamma)$. The last equality establishes the desirable result, namely that $\mathcal{M}, \Gamma \models_w \Box(\forall x)F(x)$. ■

The following observations are now easy corollaries.

OBSERVATION 6.4. *A regular and finite constant domain neighborhood frame is a* quasi-filter *if and only if every model based on it is one in which the Barcan and Converse Barcan schemas are valid.*

And since filters can be characterized as non-empty quasi-filters, and a regular CDN frame is non-empty — because regularity entails consistency.

OBSERVATION 6.5. *A regular and finite constant domain neighborhood frame is a filter if and only if every model based on it is one in which the Barcan and Converse Barcan schemas are valid.*

Now, since every filter containing finitely many worlds is augmented we have:

OBSERVATION 6.6. *A regular and finite constant domain neighborhood frame is augmented if and only if every model based on it is one in which the Barcan and Converse Barcan schemas are valid.*

For most applications this result is of some significance, even when it only applies to finite models. In section 6.3 we will offer a stronger result for infinite models. On the other hand it is easy to see that the following result holds without restrictions.

OBSERVATION 6.7. *If a constant domain neighborhood frame is augmented then the Barcan and Converse Barcan schemas are valid in all models based on it.*

6.3. Barcan formulas, epistemic possibility and duality

We remind the reader that, all formulas of the following form are Barcan formulas:

$$(\forall x)\Box\phi \to \Box(\forall x)\phi \qquad \text{(B-schema-}\Box\text{)}$$

$$\Diamond(\exists x)\phi \to (\exists x)\Diamond\phi \qquad \text{(B-schema-}\Diamond\text{)}$$

If we focus on instances of Barcan schemas where ϕ is replaced by a monadic predicate F, we obtain the following formulas:

$$(\forall x)\Box F(x) \to \Box(\forall x)F(x) \qquad \text{(Barcan-}\Box\text{)}$$

$$\Diamond(\exists x)\neg F(x) \to (\exists x)\Diamond\neg F(x) \qquad \text{(Barcan-}\Diamond\text{)}$$

These two sorts of formulas are logically equivalent in first order Kripke semantics. They are also equivalent in the (classical) extended version of neighborhood semantics presented in the previous section. The $\Box$-operator and the $\Diamond$-operator are duals in this extension. And, of course, the universal and existential quantifiers are duals as well. This is enough to guarantee the logical equivalence of Barcan-$\Box$ and Barcan-$\Diamond$.

So, all results formulated in terms of Barcan-$\Box$ can be equivalently formulated in terms of Barcan-$\Diamond$. But we argued above that the $\Box/\Diamond$ duality in standard neighborhood models is purchased at a high cost. In fact, the definition of possibility guaranteeing the duality seems inappropriate for many applications.

We proposed above a different definition of possibility (Poss) and we explained that only one half of the biconditional expressing the $\Box/\Diamond$ duality is preserved when this definition is adopted. This asymmetry would be carried to the first order case if Defintion 4.1 were adopted. In fact, we will show immediately that neither version of Barcan entails the other if (Poss) is adopted.

Counterexample. Consider a regular model $\mathcal{M}$ such that $W = \{\Gamma, \Delta, \Omega\}$, $D = (a, b)$ and $N(\Gamma) = \{\{\Gamma\}, \{\Delta\}\}$. Assume also that $\mathcal{I}(F, \Gamma) = \{\langle a\rangle\}$, $\mathcal{I}(F, \Delta) = \{\langle b\rangle\}$, and $\mathcal{I}(F, \Omega) = \{\langle b\rangle, \langle a\rangle\}$. Notice that in this case

$$\mathcal{M}, \Gamma \not\models_v (\forall x)\Box F(x)$$

in fact, for no x-variant w of v we have $|F(x)|^{\mathcal{M},w}$ in $N(\Gamma)$. Therefore we have that:

$$\mathcal{M}, \Gamma \models_v (\forall x)\Box F(x) \to \Box(\forall x)F(x)$$

Consider now $\mathcal{M}, \Gamma \models_v \Diamond(\exists x)\neg F(x)$. $|(\exists x)\neg F(x)|^{\mathcal{M},v} = \{\Gamma, \Delta\}$ is compatible with all members of $N(\Gamma)$, making true the antecedent of Barcan-$\Diamond$. But $\mathcal{M}, \Gamma \not\models_v (\exists x)\Diamond\neg F(x)$. In fact, either $|\neg F(x)|^{\mathcal{M},w}$ is $\{\Gamma\}$, when $w(x) = b$; or $|\neg F(x)|^{\mathcal{M},w}$ is $\{\Delta\}$, when $w(x) = a$. Therefore $|\neg F(x)|^{\mathcal{M},w}$ is never compatible with *all* members of $N(\Gamma)$.

The entailment from Barcan-$\Diamond$ to Barcan-$\Box$ is also severed. The following counterexample deals with this case:

Counterexample. Consider a regular model $\mathcal{M}$ such that $W = \{\Gamma, \Delta\}$, $D = (a, b)$ and $N(\Gamma) = \{\{\Gamma\}, \{\Delta\}\}$. Assume also that $\mathcal{I}(F, \Gamma) = \{\langle a\rangle\}$ and let $\mathcal{I}(F, \Delta)$ be empty. Set also $v(x) = a$.

$$\mathcal{M}, \Gamma \models_v (\forall x)\Box F(x)\,.$$

This is so in virtue of the fact that $|F(x)|^{\mathcal{M},v} = \{\Gamma\}$. But $\mathcal{M}, \Gamma \not\models_v \Box(\forall x)F(x)$. This is so because $|(\forall x)F(x)|^{\mathcal{M},v} = \{\Gamma \in W : \Gamma \models_v (\forall x)F(x)\}$ is empty. Nevertheless we have that $\mathcal{M}, \Gamma \models_v (\exists x)\Diamond\neg F(x)$. In fact, for the x-variant w of v, assigning b to x, we have that $|\neg F(x)|^{\mathcal{M},w} = \{\Gamma, \Delta\}$, which is compatible with all members of the neighborhood.

Of course the main theorems proved about necessary and sufficient conditions for augmentation of regular and finite frames continue to hold. They are formulated for the schemas $(\forall x)\Box\phi \rightarrow \Box(\forall x)\phi$ and $\Box(\forall x)\phi \rightarrow (\forall x)\Box\phi$ and nothing about the notion of possibility has been assumed in the proofs of the main observations presented above. It is also possible to show that closure under infinite intersections of a neighborhood is characterizable in terms of the validity of B-schema-$\Diamond$ for models of certain type. We will proceed to show this fact now.

A frame $\langle W, N, D\rangle$ is *monotonic* if and only if the number of objects in its domain is as large as the number of sets in the neighborhood with the largest number of sets in the model. The intuitive idea behind monotonicity is that an agent represented by a monotonic model is capable of discerning at least as many objects in the universe as beliefs he is capable of entertaining.

OBSERVATION 6.8. *A monotonic constant domain neighborhood frame is closed under infinite intersections if and only if every model based on it is one in which B-schema-$\Diamond$ is valid.*

PROOF. First we will show that if $\langle W, N, D\rangle$ is a *constant domain neighborhood* frame then:

$$\text{If } \bigcap N(\Gamma) \in N(\Gamma) \text{ then } \mathcal{M}, \Gamma \models_w \Diamond(\exists x)F(x) \rightarrow (\exists x)\Diamond F(x).$$

Assume both closure under arbitrary intersections and $\mathcal{M}, \Gamma \models_w \Diamond(\exists x)F(x)$. Therefore $|(\exists x)F(x)|^{\mathcal{M},w} \cap X \neq \emptyset$, for every X in $N(\Gamma)$. Now, since we are assuming closure under intersections, we have that:

$$|(\exists x)F(x)|^{\mathcal{M},w} \bigcap(\bigcap N(\Gamma)) \neq \emptyset$$

Therefore we know that there is Δ in $\bigcap N(\Gamma)$ such that:

$$\mathcal{M}, \Delta \models_w (\exists x)F(x)\,.$$

This means that there is a x-variant v of w, such that: $\mathcal{M}, \Delta \models_v F(x)$. So, $\Delta \in |F(x)|^{\mathcal{M},v} = \{\Sigma : \mathcal{M}, \Sigma \models_v F(x)\}$. Therefore:

$$|F(x)|^{\mathcal{M},v} \cap X \neq \emptyset, \text{ for every } X \text{ in } N(\Gamma).$$

This indicates that there is a x-variant v of w such that:

$$\mathcal{M}, \Gamma \models_v \Diamond F(x) .$$

which, in turn, guarantees that:

$$\mathcal{M}, \Gamma \models_w (\exists x)\Diamond F(x) .$$

Monotonicity is not needed in order to establish the previous result. The property will, nevertheless, be needed in order to establish the converse. As in previous observations we will assume that a monotonic frame is not closed under finite intersections. Then we will construct a family of models based on such frame where a particular instance of Barcan-schema-$\Diamond$ is not valid.

Since $\bigcap N(\Gamma) \notin N(\Gamma)$, we can select a world Δ^X in $X - \bigcap N(\Gamma)$ for every $X \in N(\Gamma)$, making sure that $\Delta^X \neq \Delta^Y$ for $X \neq Y$. Select now for each Δ^X an element c^X of D, in such a manner that $c^X \neq c^Y$ for $\Delta^X \neq \Delta^Y$. Monotonicity guarantees that there are enough constants in D in order to make the previous selection.

Let now $\langle c^X \rangle \in \mathcal{I}(F, \Delta^X)$ for each X in $N(\Gamma)$ and make sure there is no other c in D, such that $\langle c \rangle \in \mathcal{I}(F, \Delta^X)$, for each Δ^X in W. Let, in addition, $\mathcal{I}(F, \Delta)$ be empty for any world Δ different from some Δ^X in W.

Now notice that for an arbitrary valuation w:

$$\mathcal{M}, \Gamma \models_w \Diamond(\exists x)F(x)$$

To see this notice that the set $|(\exists x)F(x)|^{\mathcal{M},w}$ contains all the worlds Δ^X for each X in $N(\Gamma)$. In fact, we know that it is always possible to find a x-variant v of w such that, for an arbitrarily chosen world Δ^X:

$$\mathcal{M}, \Delta^X \models_v F(x) .$$

the valuation v is the one assigning c^X to the variable x. Therefore the set $|(\exists x)F(x)|^{\mathcal{M},w}$ intersects all sets X in $N(\Gamma)$. Nevertheless:

$$\mathcal{M}, \Gamma \not\models_w (\exists x)\Diamond F(x) .$$

This is so in virtue of the fact that there is no x-variant v of w such that $|F(x)|^{\mathcal{M},v}$ is compatible with all sets in $N(\Gamma)$. ∎

Now it is simple to establish the following observation, characterizing augmentation for infinite models.

Observation 6.9. *A regular and monotonic constant domain neighborhood frame is augmented if and only if every model based on it is one in which the Barcan-schema-$\Diamond$ and Converse Barcan schemas are valid.*

6.4. The role of the Barcan formulas in first order epistemic semantics

The previous results shed some light on a family of interesting issues. Let's focus first on consequences for likelihood operators. In the previous sections we began to articulate some of the interesting relationships between lack of logical omniscience and likelihood, but this analysis depended on the limited expressive power of the (modal) propositional language.

Definition 6.9. A neighborhood $N(\Gamma)$ is *weakly inconsistent* if and only if it is consistent (the empty set is not in the neighborhood) but $\bigcap N(\Gamma)$ is empty.

Definition 6.10. A constant domain neighborhood frame $\langle W, N, D\rangle$ is *weakly consistent* if and only if it does not contain weakly inconsistent neighborhoods.

One of the advantages of neighborhood semantics is its capacity to encode weakly inconsistent neighborhoods. Let's consider again the 'lottery' examples briefly reviewed when we dealt with the notion of epistemic possibility. Say that you consider a lottery with a large number of tickets. Focus, in adddition, on a particular neighborhood $N(\Gamma)$. Say that at his world the represented agent considers highly likely that each ticket in the lottery will not win. This gives us a consistent neighborhood which is also weakly consistent. Nevertheless, in this situation the agent might also consider highly likely that some ticket will win. Now the neighborhood is weakly inconsistent. Situations of this sort are common if one wants to have a logic of 'high likelihood'.

Examples of this sort give a prima facie indication of the inadequacy of representations of the notion of likelihood using the $\Box$ operator within the framework of Kripke semantics. To be more precise what is clearly inadequate is to use a $\Box$ operator with its usual semantic characterization via constant domain Kripke frames. The reason is that such frames are pointwise equivalent to augmented neighborhood frames. And, according to the results presented in the previous section, this means that the corresponding neighborhood frames *cannot* contain weakly inconsistent neighborhoods.

Things could be made clearer by noticing that the Barcan formula admits an interpretation encoding the lottery paradox itself. In fact, assign the interpretation 'ticket x does not win' to '$F(x)$', in the Barcan formula. Assume also that the $\Box$ operator in the Barcan formula is interpreted as a 'high likelihood' operator. Then the intended meaning of the Barcan formula carries the content of the paradox itself. It says that if every ticket is highly likely to be a loser then it is highly likely that the lottery has no winning ticket. If one attempts to represent the notion of likelihood via a $\Box$ operator and the operator is semantically characterized via regular frames with constant domains, this leads to a clearly undesirable situation. Of course, this applies to *any* $\Box$ operator characterized via its classical truth conditions in constant domains Kripke frames.

Of course, there might be another manner of characterizing the notion of likelihood in constant domain Kripke frames. Some authors have proposed different truth conditions (see (Halpern-Rabin, 1987)). Nevertheless the truth conditions already explored in the literature seem equally inadequate for different reasons. Halpern and Rabin, for example, proposed using an operator sharing important features with a classical $\Diamond$ operator. We will comment on this alternative below.

Finally one can appeal to Krikpe frames with varying domains. In this case the Barcan and Converse Barcan formulas are not necessarily validated. Perhaps this can be developed as a principled solution. Nevertheless, constant domain Montague-Scott models can indeed be used to represent (quite naturally) likelihood operators via a $\Box$ operator endowed with its classical truth-definition (in neighborhood semantics). Perhaps there are reasons for representing some epistemic operators by appealing to varying domains. It is less clear that such types of first order structure were required for the representation of all types of likelihood operators.

Summing up. We showed that the validity of both the Barcan and Converse Barcan schemas correspond to interesting conditions on neighborhood frames. With some provisos they provide necessary and sufficient conditions for the formation of augmented neighborhoods. The constancy of the domain does not force, however, the validity of the formulas. There are constant domain neighborhood frames where the formulas fail to be validated. And this is beneficial given that in this setting the formulas might carry the paradoxical content of the so-called lottery paradox.

We showed that regular neighborhood frames can be naturally used in order to represent a monadic operator of likelihood. Moreover this can be done by using (a conveniently constrained) standard $\Box$ operator. It should be also be said that, since augmented frames are pointwise equivalent to Kripke frames, the Barcan and Converse Barcan formulas provide conditions for representing Kripke frames as neighborhood frames. Therefore the formulas provide interesting conditions for comparing both semantics.[12]

[12] The abstract issue of what operators should be considered modal operators is difficult to delimit. There are nevertheless some interesting attempts to do so. The *structuralist*

A final remark about the use of first order neighborhood semantics in order to represent other epistemic notions (aside from likelihood or high likelihood). Many authors have argued on normative grounds that the representation of notions like certainty and knowledge *require* logical omniscience. This has been articulated in different manners in terms of the internal consistency of the represented agent.

Suppose a man says to you: 'I know that p but I do not know whether q'.[13] To what extent is he inconsistent if q does follow (logically) from p? Hintikka has argued that in this case the agent is not necessarily inconsistent in any psychological or quasi-psychological sense of the word. But, according to his terminology the agent has put himself in an *indefensible* position. I.e. a position that is not *immune to certain kind of criticism.* So, he proposed using the notions of *defensibility* and *indefensibility* in epistemic logic, rather than the notions of consistency and inconsistency.

Arguing from a different point of view, Isaac Levi has proposed to see (logically closed) belief sets as representing not our explicit beliefs, but our *commitments* to belief or knowledge (see (Levi, 1980)). When the person described in the previous situation says: 'I know that p but I do not know whether q', she is not aware of what she is committed to know. She should be capable of self-correction by the mere articulation of her commitments.

This is not the place to discuss these issues in detail. It is enough to observe that a theoretical commitment to represent belief (knowledge) via closed and consistent bodies of sentences is tantamount in our setting to be committed to the validity of the Barcan and Converse Barcan formulas. In fact, regular frames containing augmented neighborhoods are sufficient to determine the validity of these formulas. This, nevertheless, is not a problem in our setting. In fact, the validity of the Barcan and Converse Barcan formulas for other epistemic notions beside likelihood is clearly unproblem-

account offered in (Koslow, 1992) is one of these attempts. According to this account the modal operators determined by sub-augmented (classical) models are not modal operators. Perhaps this view is too stringent. On the one hand under a pre-systematic point of view the notion of 'probable enough' seems to be a modal notion. On the other hand the techniques used both by the structuralist and the algebraic accounts of modality seem to be 'close neighbors' of the Montague-Scott approach. One can conjecture that some of the results presented here do have interesting structuralist counterparts. So, perhaps Koslow's account of modality can be generalized. Otherwise, a reformulated version of our results can perhaps be used in this setting in order to state the necessary and sufficient conditions (in quantified models) that a 'classical' operator should satisfy in order to be a modal operator (in Koslow's sense).

[13] See (Hintikka, 1962), p. 31. According to the nomenclature in (Hintikka, 1962), 'p', 'q', etc. are formulas, not prpositions.

atic. If the $\Box$ operator is interpreted as a certainty operator in the lottery example presented above, we do not have a puzzling situation. If you are certain that each ticket will not win the lottery, you had better be certain that no ticket will win. The same applies to the notion of knowledge.

7. Halpern and Rabin's logic of likelihood

As we explained above Halpern and Rabin presented in (Halpern-Rabin, 1987) a Kripkean account of a monadic likelihood operator. Their proposal has many interesting features. Among others, they present a multi-modal logic, combining a likelihood operator (L) and a certainty operator (G). The paper is also a precursor of the excellent work on multi-agent systems done by Halpern in collaboration with Fagin, Moses and Vardi (Fagin et al., 1995). Some of this work includes the study of languages where explicit mention of probabilities in formulas is allowed ((Fagin-Halpern, 1994) is an example).[14] The modalities considered in these papers have enough expressivity to say 'according to agent i, formula A holds with probability of at least b'. Languages of these sort are potentially treatable in the framework presented in this paper, but here we will limit ourselves to a comparison with the purely modal language considered in (Halpern-Rabin, 1987) (further extended in (Halpern-McAllester, 1989)).

An *LL model* is a quadruple $M = (S, \mathcal{L}, C, \Pi)$, where S is a set of states, $\mathcal{L}$ and C are binary relations on S with $\mathcal{L}$ reflexive. Π is a mapping from the primitive propositional letters to 2^S. Intuitively Π is a valuation, which according to the authors, 'associates with each primitive proposition the set of states of which it is true'. This seems standard and simple, but we will see immediately that the latter intuitive description should be taken 'cum grano salis'. $\mathcal{L}$ and C are the accessibility relations used in the characterization of the operators L and G. The truth conditions for these operators provide a first intuitive idea of the intended interpretation of $\mathcal{L}$ and C.

Π is extended to a mapping from the entire set of LL formulas to 2^S. The crucial clauses of the extension are:

(G) $\Pi(\mathrm{G}p) = \{s :$ for all t reachable[15] from $s, t \in \Pi(p)\}$,

(L) $\Pi(\mathrm{L}p) = \{s :$ for some t with $(s,t) \in \mathcal{L},\ t \in \Pi(p)\}$.

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[14] (Heifetz-Mongin, 1998) also considers a similar research strategy and offers a very good overview of recent work in this area.

[15] A state t is *reachable* from s if, for some finite sequence $s_0, \ldots, s_k$, we have $s_0 = s$, $s_k = t$, and $(s_i, s_{i+1}) \in \mathcal{L} \cup C$, for $i < k$.

As usual the authors write $\mathcal{M}, s \models p$ instead $s \in \Pi(p)$. Surprisingly, and against the previously quoted interpretation of $\Pi, s \models p$ should *not* be read as 'p is true at state s'. The following quotation makes this issue clear:

> We emphasize that if s contains the formula p (i.e. if $\mathcal{M}, s \models p$), this *does not mean* that p is actually true at s, but rather that p is one of the hypotheses that we are taking to be true at this state.

Here we are facing one of the foundational problems mentioned in the introduction of this paper. Namely, the problem related to the need of encoding the epistemic states of agents as components of possible worlds. Sometimes worlds themselves are understood as idealized epistemic states of agents. I.e. worlds are understood as ways of encoding the facts taken as true by idealized agents who are not only logically omniscient but also epistemically omniscient. This is the intended interpretation of worlds in a LL-model. A state s is seen as consisting of 'a set of hypotheses that the agent takes to be 'true for now'. The authors are aware of the extreme idealization involved in their modeling.

> The assumption that s is complete is an idealization. In practice we imagine that s is a finite set consisting of all formulas 'relevant' to the discussion, and perhaps all their sub-formulas.

Although the actual formalism implements a strong form of omniscience, which is both logical and epistemic, the authors have in mind implementations which lack both forms of omniscience. On the other hand the cautionary notes about omniscience have no formal counterpart. So, for example the L operator has properties like:

$$A \rightarrow \mathrm{L}(A) \tag{Ax5}$$

Axiom 5, when interpreted literally says that true facts should be considered likely (taking likelihood as some undetermined epistemic property weaker than knowledge and certainty). This literal interpretation of the axiom is of course problematic. Nevertheless, when Ax5 is interpreted by taking into account the (informal) epistemic role played by worlds in the LL model, the axiom says something like: 'if A is taken as true by a rational agent, he should consider A likely as well'. And this interpretation seems more adequate.

Notice that if we only take into account formal definitions and we ignore completely any intended interpretation of the LL logic, then the truth definition of the L-operator coincides with the definition of (ontological) possibility in alethic modal logic.

Now, the notions of epistemic and alethic (or ontological) possibility should not be conflated. For example, Ax5 is a perfectly good axiom for the notion of ontological possibility. Under this interpretation Ax5 just says that every true event is (objectively) possible. In contrast, the axiom is inadequate for any epistemically motivated notion of possibility. The mere fact that an event is objectively true does not determine that a rational agent must judge it epistemically possible. The notion of epistemic possibility is always relative to some stock of assumed background knowledge. An event is epistemically possible if compatible with that body of assumed knowledge, quite independently of whether this corpus of assumed knowledge is or not true.

Of course, as we explained above, when we add the informal interpretation that the authors have in mind, Ax5 is less problematic both as a constraint on epistemic possibility and as a constraint on likelihood.[16] Nevertheless, as the authors themselves point out, there are other properties of the notion of alethic possibility that cannot be reconciled at all with a notion of likelihood understood as 'probability greater than a threshold'. The problem is the axiom:

$$\mathrm{L}(A \vee B) \leftrightarrow (\mathrm{L}(A) \vee \mathrm{L}(B)) \tag{Ax6}$$

Halpern and Rabin provide a counterexample and concluded that "it is inappropriate to think of L as meaning 'with probability greater or equal than one half'". The problem is easy to see. If you are waiting for a subway at 72 street you might consider highly likely that the next subway will be either a local or an express train (because you think trains are running) but you might assign probability 1/2 to the event that the next train is a local and 1/2 to the event that is a express. The authors propose to capture the notion of 'A is highly likely' via $\mathrm{LG}(A)$ — work in this direction was further pursued in (Halpern-McAllester, 1989). Nevertheless the motivation for this translation is not completely transparent, and the issue of the nature of L's intended interpretation[17] seems to remain unresolved.

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[16] Some of the latter work in this area by Halpern and Fagin seem to implement formally restrictions which in (Halpern-Rabin, 1987) are only informally made. (Fagin-Halpern, 1994), for example, contains formulas of the type $w_i(A) \geq b$, with the meaning 'according to agent i, formula A holds with probability of at least b'. Axiom (**W9**), $A \Rightarrow (w_i(A) = 1)$, is then relativized is such a way that it holds only when A is an i-probability formula or the negation of an an i-probability formula. If $w_i(A) = 1$ is then informally understood as stating: 'agent i is *almost certain* about A', then the axiom states that agents should be almost certain about their probability judgements — rather than almost certain about A, if A is true.

[17] In (Halpern-McAllester, 1989) the English statement 'The coin is likely to land heads twice in a row' = C, is translated as 'It is a likely to be necessarily the case that the

Summing up. The authors start with an (informal) epistemic interpretation of worlds as sets of assumed hypotheses of agents who are both epistemically and logically omniscient. Then an L-successor of any state s (under the corresponding $\mathcal{L}$ relation) describes a set of hypotheses that the agent judges as reasonably likely. C-successors of s, in contrast, are points that are judged as conceivable from the point of view of s, but not necessarily reasonably likely. Then the sentence $\mathrm{L}(A)$, saying that A is likely, is true at s whenever there is a $\mathcal{L}$-reachable state t from s where A holds true. $\mathrm{G}(A)$ holds true at s when A is true at all states reachable from s. In other words, $\mathrm{G}(A)$ holds true at s when A is true at all states deemed as conceivable from the point of view of s.[18]

The proposal has some problems, which we summarized above. Encoding epistemic states as worlds is too strong an idealization and the L-operator shares too many properties with the alethic notion of possibility to reflect qualitatively a probabilistic construal of likelihood. Perhaps the L-operator can be used more naturally in order to reflect the notion of '(epistemically) possible to a sufficient degree' in the sense of (Dubois-Prade, 1992) (see footnote 6 above).

Constructions in terms of neighborhood semantics seem more appropriate, in contrast, to capture several epistemic notions, like 'probable to a sufficient degree'. In the proposed framework probabilistic operators can be reflected via the use of single $\Box$ operators in appropriate models. In contrast, we have seen that the best efforts to encode probabilistic operators in the Kripkean framework requires appealing to combinations of modal operators, where the intended interpretation of some of the components used in the translation remains problematic. Neighborhood constructions circumvent the problem of epistemic omniscience by neatly separating what is true, i.e. what holds true at a world, from what is *judged as true* by an agent, i.e. the propositions in the neighborhood of that world. In addition,

coins lands heads twice in a row' $= \mathrm{LG}(C)$ — not as $\mathrm{L}(C)$. The justification for this translation rests, not on intuition, but on the formal fact (proved by the authors) that inferences made in LL are *sound* with respect to the translation. "This means that if we have a set of probability assertions about a certain domain, and translate them (using the suggested translation) into LL, and then reason in LL, any conclusions we draw will be true when interpreted as probability assertions about the domain." (Halpern-McAllester, 1989, p. 137) In contrast, translating the English statement C directly as $\mathrm{L}(C)$, leads to inconsistencies. Therefore the notion of 'reasonably likely' encoded by L should be other than 'holds with probability greater than a threshold'. Aside from this negative result the authors do not provide further guidance as to the intended meaning of L.

[18] According to the authors the conceivability relation should be viewed as the reflexive transition closure of $\mathcal{L} \cup C$.

logical omniscience is also circumvented because the neighborhoods need not be augmented. Moreover, as Kyburg has pointed out in several occasions, the development of a logic of likelihood seems to require the ability of encoding weakly inconsistent scenarios. Also, as Kyburg pointed out in (Kyburg, 1995) the latter fact does not seem to require the use of some form of *paraconsistency*. The underlying notion of logical consequence in all the constructions studied in this paper is perfectly classical.

Acknowledgements. I am grateful to Henry E. Kyburg Jr., Ruth Barcan Marcus, Rohit Parikh and two anonymous referees for helpful comments.

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