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Five Contributions to Natural Philosophy

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Five Contributions to

Natural Philosophy

by Walter Noll

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dedicated to the memory of
Clifford Truesdell, 1919 - 2000

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[0] Introduction.

In the 17th and 18th century, mathematics and physics were not the separate specialties that they are today, and *Natural Philosophy* was the term used for the endeavor to understand nature by using conceptual mathematical tools. Perhaps the most important scientific work of that era is Newton's *Philosophiae Naturalis Principia Mathematica*, published in 1687. He invented differential calculus, with new mathematical concepts, a new terminology, and new notations, which made possible a concise formulation of the laws of particle mechanics now named after him.*)

In the 1960's, Clifford Truesdell tried, and to some extent succeeded, to revive not only the term *Natural Philosophy* but also the spirit behind it. He was the driving force behind the founding, in 1963, of the *Society for Natural Philosophy*, which still exists today. (I was one of the founding members.) Much of my own work has been in the spirit of Natural Philosophy. In recent years, many of the mathematicians working in the field of continuum physics have de-emphasized this spirit and have concentrated more and more on obtaining existence theorems and other "results" rather than analyzing basic concepts. I have been out of step with this trend, but I have not been completely idle. This collection presents some of the ideas that I have developed in the past 10 years or so.

In my doctoral thesis in 1954, I discussed a general principle which I called "The Principle of Isotropy of Space". My thesis advisor, Clifford Truesdell, teased me by putting up his son, then about 10 years old, to ask me: "Mr. Noll, please explain the Principle of Isotropy of Space to me." It is only recently, more than 40 years later, that I have found a good explanation, namely the one presented in the first two papers [1] and [2] of this collection. (In 1958, I realized that "Principle of Isotropy of Space" is misleading and renamed it the "Principle of Objectivity". Later, Clifford Truesdell and I decided that this term is also unsatisfactory and we settled on "Principle of Material Frame-Indifference". See the exchange of letters reproduced on pp.28 -29 of [N6] and pp. XI and XII of the third edition of [NLFT]).

The treatise entitled *The Non-Linear Field Theories of Mechanics* [NLFT] by C. Truesdell and me appeared in 1965 as a part of the *Encyclopeadia of Physics*. It has become a standard reference in its field. A second edition appeared in 1992, which was translated into Chinese in 2000. A third edition came out in 2004. All these editions are just reprints of the first edition, except for some minor corrections and new prefaces, one in the second edition by Clifford Truesdell and me, and one in the third edition by Stuart Antman. The last edition also contains an excerpt from [N6]. Already in the 1970s it had become

*) However, his terminology and notation are different from those that are common now. He used "fluxion" instead of "derivative" and denoted it by superimposed dots. (In his honor, I use a superscript thick dot instead of the more common superscript prime.)

clear to me that [NLFT] had some serious flaws. They were alluded to in the preface of the second edition. Here is an excerpt from that preface:

" Many of the concepts, terms, and notations we introduced have become more or less standard, and thus communication among researchers in the field has been eased. On the other hand, some ill-chosen terms are still current. Examples are the use of 'configuration' and 'deformation' for what we should have called, and now call, 'placement' and 'transplacement', respectively. (To classify translations and rotations as deformations clashes too severely with the dictionary meaning of the latter.) ...On p.12 of the Introduction (of the first edition) we stated '... we have subordinated detail to importance and, above all, clarity and finality'. We believe now that finality is much more elusive than it seemed at the time. The *General theory of material behavior* presented in Chapter C, although still useful, can no longer be regarded as the final word. The Principle of Determinism for the Stress stated on p.56 has only limited scope. It should be replaced by a more inclusive principle, using the concept of *state* rather than a history of infinite duration, as a basic ingredient. In fact, forcing the theory of *materials of the rate type* into the general framework of the treatise as is done on p. 95 must now be regarded as artificial at best, and unworkable in general. This difficulty was alluded to in footnote 1 on p.98 and in the discussion of B. Bernstein's concept of a *material* on p.405. This major conceptual issue was first resolved in 1972 (in [N7]), and then only for simple materials."

I believe that [NLFT] is now in many respects obsolete and should be updated. Such an update would be an enormous job. I am too old to get involved in it and Clifford Truesdell is no longer with us. However, in the third paper [3] of this collection I present some guidelines for such an update. A fundamental ingredient should be the concept of a *State-Space System*, which is developed in detail in [3].

In the forth paper [4] of this collection I use the ideas presented in [3] and summarize some of the content of [N7] in order to introduce the concept of a *Simple Semi-Liquid*, which can furnish a conceptual infrastructure for a significant part of what is known as *Rheology*.

The last paper [5] of this collection is a sequel to [4] and deals with a special case that may serve as a mathematical model for the flow of nematic liquid crystals. I call it the theory of Nematic Semi-Liquids. Applying the results of the general theory to this special case gives many interesting formulas, some of which were first obtained by J.L.Ericksen in 1960 using an entirely different context.

The papers [4] and [5] could serve as a basis for updating Chapter E of [NLFT].

None of the papers in this collection have been published before, but preliminary versions of [2] and [4] were written some time ago. In particular, an earlier version of [2] was submitted to the *Reviews of Modern Physics* in 1995. I thought that this paper should be of interest to an audience wider than just

those interested only in the mathematics of continuum physics. A letter from the editor of the *Reviews*, written in 1988, informed me that a 1961 paper by Bernard Coleman and me had become a *citation classic* and that they would welcome receiving other papers from me. However, my paper was rejected even though the editor conceded that "the article is clearly written". Here are some quotes from the reviewer:

"I enjoyed reading this paper and very much would like to see it published. I am afraid, however, that the Reviews of Modern Physics is not the appropriate place. I believe that the overwhelming majority of the readers of the journal will consider the paper unreadable. Not because the material presented is intrinsically difficult, but rather because the author's individual form of the 'Bourbakian' style is far removed from anything that physicists are willing to digest. .... Professor Noll is highly respected in the mathematical community and has more than once proved himself to be ahead of his time. ..."

This experience and others like it led me to the conclusion that it would be frustrating and unwise to submit any of my work to standard journals. In the meantime, technology made it easier to do my own typesetting, and I decided to bring out this collection by myself.

I have presented many of the ideas in this collection in a variety of lectures, most of them given in Italy in the Fall of 2000 and in November 2002. The paper [1] is based on a lecture I gave on October 30, 1997, at the Center for Philosophy of Science of the University of Pittsburgh. A preliminary version of [N2] was written in 1995. The paper [3] is based on lectures I gave in Turin, Italy, in November 2002 and at the University of Illinois in March 2003. A preliminary version of [N4] was written in 1996. The paper [5] was written in 2002.

Acknowledgements:

Much of the work leading to these papers was done while I was a visiting professor in Italy, for a month each in Pisa in 1993 and 1996 and in Pavia in 2000. I am grateful for the help and encouragement from my friend Epifanio Virga, now at the University of Pavia, and for the financial support from the Italian Ministry of Education.

I am also grateful to Millard Beatty, Victor Mizel, Ian Murdoch, and my wife Marilyn for proofreading help and valuable suggestions.

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[1]-[5] : See table of contents.

[NLFT] : Truesdell.C. and Noll,W. , *The Non-Linear Field Theories of Mechanics*, third edition, Springer-Verlag, 2004.

[N6] : Noll,W., *The Genesis of the Non-Linear Field Theories of Mechanics*, Journal of Elasticity **70**, 23-30, 2003. Also posted on the website *math.cmu.edu/∼ wn0g/noll* .

[N7] : Noll,W., *A New Mathematical Theory of Simple Materials* , Archive for Rational Mechanics and Analysis **48**, 1-50, 1972.

[1] On the Illusion of Physical Space

.

1. Physical space and frames of reference.

My dictionary *) gives 19 definitions for "space". We consider only the first of these, vague and ambiguous as it may be:

"The unlimited and indefinitely three-dimensional expanse in which all material objects are located and all events occur." When talking about *physical space*, we mean *space* as implied by this definition.

Many people, scientists or not, use the words "space", "space and time", "point in space", "position", "location", "motion", and "speed" as if they have an unambiguous and obvious meaning regardless of context. Physicists often do the same with related terms such as "velocity", "acceleration", "momentum", and "kinetic energy". However, it takes very little reflection to realize that it makes no sense to speak of *location* and hence of *motion* etc. except relative to an explicitly or tacitly specified **frame of reference** . To this day, physicists seem to be brainwashed by Newton's idea of *absolute space*. He distinguishes it from what he calls "relative space", which is what I now call a "frame of reference". He states (in *Philosophiae Naturalis Principia Mathematica*, 1687):

"Absolute space, in its own nature, without relation to anything external, remains always similar and immovable."(translation by Andrew Motte)

Newton then also distinguishes between absolute and relative positions and absolute and relative motions.

Only about 20 years after publication of Newton's treatise, George Berkeley (in *The Principles of Human Knowledge*, 1710) already realized that Newton's idea of absolute space made no sense:

"But notwithstanding what has been said, it does not appear to me that there can be any motion other than relative."

Much later, Ernst Mach (in his *Die Mechanik in Ihrer Entwicklung*, 1883), came to the same conclusion:

"But if we take our stand in the basis of facts, we shall find that we have knowledge only of relative spaces and motions. Relatively, ..., the motions of the universe are the same whether we adopt the Ptolemaic or Copernican mode of view. Both views are, indeed, equally correct; only the latter is more simple and more practical. The universe is not twice given, with an earth at rest and an earth in motion; but only once, with its relative motions alone determinable." (translation by E. J. McCormack)

In the late 1950s, I formulated what is now called the *Principle of Material Frame-Indifference*. It essentially states that physical laws should not

*) The Random House College Dictionary

depend on whatever frame of reference is used to describe them. This principle is not a law of physics, it is merely a guide to prevent the nonsensical formulation of alleged physical laws. Nevertheless, the consequences of this principle are not entirely trivial. In 1996, in a preliminary version of the paper [$N2$] following this one, I wrote:

"Most people, whether they are physicists or not, use the words 'position', 'place', and 'motion' as if they had a completely unambiguous meaning. It should be obvious, however, that it makes no sense to speak of the position or location of a material particle or point except relative to a *frame of reference*. Similarly, *motion* means change of position with time and hence, like *position*, makes no sense except relative to a frame of reference. Usually, it is tacitly understood that the frame of reference to be used is the one that is determined by the background. In our daily lives, the background is most often rigidly attached to the earth (the road, the building we live or work in, the scenery, etc.). Sometimes, for example when we are inside a railroad car, ship, airplane, or spacecraft, the background is given by the interior walls of a conveyance that is in motion relative to the frame of reference provided by the earth. When our ancestors looked at the sky at night, they saw the background provided by the fixed stars and used the frame of reference determined by it to describe the motion of the planets (recall that 'planet' is the Greek word for wanderer).

We all know about the trouble that Galileo had when he asserted that the earth moves around the sun rather than the sun around the earth, as church dogma had it at that time. On the face of it, neither of these assertions makes any sense because frames of reference are not specified. Both of these assertions are in fact true if 'move' is understood relative to different but suitable frames of reference. I cannot understand, therefore, what the fuss was all about.

When describing a physical process, there is sometimes no obvious background that can be used to determine a frame of reference. Even if there is, the frame obtained from it may not lead to the simplest description of the process and a simpler description may be obtained by using a frame that seems, at first, artificial. The true value of the Copernican frame, although quite artificial at the time when it was proposed, is that it yields simpler motions for the planets than the frame determined by the earth or the frame determined by the fixed stars and the condition that the center of the earth be at rest. The choice of a frame of reference is a matter of expediency, not of truth.

As stated above, a frame of reference should make it possible to speak of *locations*. Mathematically, locations are points in a (genuine) *Euclidean space* (as defined precisely in Chapter 4 of my book [FDS]) which we will then call a *frame-space*. Such a frame-space can be constructed from a suitable rigid material system (the most important example is the earth). ... Only when such a frame-space is at hand does it make any sense to use geometric concepts such as *straight line, direction, angle*, etc. and also to talk about *vectors*, namely as members of the translation-space of the frame-space. Hence concepts such as *velocity, acceleration*, and *force* also require the specification of a frame-space.

Some people confuse the concept of a frame of reference with that of a

coordinate system. It makes no sense to talk about a coordinate system unless a frame-space (or at least some kind of manifold) is given first. One can consider many different coordinate systems on one and the same frame-space. Using coordinate systems in conceptual considerations is an impediment to insight; they have a legitimate place only in the context of very specific situations."

The technical details of the construction of a frame-space from a rigid material system are not entirely trivial and are given in the paper [N2] that follows.

In 1997, the psychologist and neuro-scientist Steven Pinker, in his book [P], has a section entitled *Frames of Reference*, in which he writes:

"Reference frames are inextricable from the very idea of location. How do you answer the question 'Where is it?' By naming an object that the asker already knows - the frame of reference - and describing how far and in what direction the 'it' is, relative to the frame. A description in words like 'next to the fridge', a street address, compass directions, latitude and longitude, Global Positioning System satellite coordinates - they all indicate distance and direction relative to a reference frame."

2. The origin of the illusion.

It should be clear from the above that the existence of a physical space, apart from any frame of reference, is an illusion. The mystery is that so many intelligent people, Newton included, fell victim to this illusion. Immanuel Kant (*Critique of Pure Reason, 1781*) was another such victim:

"Space is not an empirical concept that can be abstracted from external experience. Space is a necessary a priori conception, which is at the basis of all external visualizations. It is impossible to imagine that there is no space, even though one can think of it as being devoid of any objects. Therefore, space must be considered as condition for the possibility of real phenomena and not dependent on these." (my translation)

Despite the fact that I was aware of Mach's ideas early in my career, I was also a victim of the illusion of physical space until the early nineteen sixties. In my doctoral thesis, in 1954, I proposed what I called "the principle of isotropy of space". Later I realized that there is no such thing as physical space and that the "the principle of isotropy of space" should be replaced by what is now called the *principle of frame-indifference*.

I submit that a solution of the mystery of the illusion of physical space does not come from physics but from psychology and neural science, and in particular from the way our brain processes visual information. A brilliant analysis of this way is given in the Chapter entitled *The Minds Eye* in the Book [P] by Steven Pinker. Here are some quotes:

"I think stereo vision is one of the glories of nature and a paradigm of how other parts of the mind might work. Stereo vision is information processing

that we experience as a particular flavor of consciousness, a connection between mental computation and awareness that is so lawful that computer programmers can manipulate it to enchant millions. It is a module in several senses: it works without the rest of the mind (not needing recognizable objects), the rest of the mind works without it (getting by, if it has to, with other depth analyzers), it imposes particular demands on the wiring of the brain, and it depends on principles specific to its problem (the geometry of binocular parallax). Though stereo vision develops in childhood and is sensitive to experience, it is not insightfully described as learned or as a mixture of nature and nurture; the development is part of an assembly schedule and the sensitivity to experience is a circumscribed intake of information by a structured system. The key to using visual information is not to remold it but *access* it properly, and that calls for a useful reference frame."

Thus, it seems that the predisposition to fixate on a particular frame of reference at any given situation is hardwired into our brain at birth, just as is the ability to acquire language. Which particular frame we fixate on (or which particular language we learn) depends on the environment. Usually, it is the background that determines this fixation. When we talk about motion we mean motion relative to the fixated frame, without being consciously aware that we do so. Our brain chooses the fixation of the frame of reference in such a way that it facilitates our ability to understand our environment with as little mental computation as possible. Occasionally, we may fixate on a frame that is less than appropriate. Most of us had an experience like the following:

(1) We lie flat on the grass looking up into the sky and see a few white clouds. We also see the top of a lone tree. Suddenly, it appears that the tree is slowly tipping over while in reality, relative to the earth, the clouds are slowly moving. The fixation has shifted from the frame of reference determined by the earth to a frame of reference determined by the clouds.

(2) We sit in a train at rest at a station and look at another train on an adjacent track. Suddenly, our train seems to start slowly moving while in reality, relative to the tracks, we are still at rest but the other train is moving. The fixation has shifted from the frame of reference determined by the platform of the station to the frame of reference determined by the other train.

A recent article entitled *Weightlessness and the Human Body* by Ronald H. White (Scientific American, September 1998) describes how astronauts sometimes become victims of fixation on an inappropriate frame:

(a) "When space travelers grasp the wall of their spacecraft and pull and push their bodies back and forth, they say it feels as though they are stationary and the spacecraft is moving".

(b) "Returning space travelers report experiencing a variety of illusions - for example, during head motions it is their surroundings that seem to be moving - and they wobble while trying to stand straight..."

Notice that in all of the examples given above, our brain does not merely perceive the relative motion, but it makes a decision on what is fixed

and what is in motion. The following example may indicate why this is useful: Suppose we had to deal with four objects in relative motion to one another. It would be more difficult for the brain to keep track of the resulting six relative motions than to fixate on one object - the one most prominent - as a basis for a frame of reference and just analyze the motions of the three remaining objects relative to this frame.

The fixation on a frame is an involuntary unconscious process, and hence we may fall victim to the illusion that the frame is independent of the presence of any objects around us and hence becomes "physical space". In a way, we have been brain-washed by our own brains into the belief in a physical space.

One must recognize, however, that Newton had also a physical reason for introducing the idea of absolute space. He may have believed that he needed it to account for the phenomenon of inertia. The law $\mathbf{f} = m\mathbf{a}$, which is known to everybody as *Newton's law*, involves the acceleration $\mathbf{a}$. But acceleration is meaningful only relative to a given frame of reference, and Newton's law cannot be valid if this frame of reference is arbitrary. Hence Newton may have introduced his *absolute space* to be a frame of reference in which his law holds. However, Newton's law is then valid also in any frame that moves uniformly with constant velocity relative to absolute space. Therefore, there are infinitely many frames in which Newton's law holds. We now call such frames **inertial frames**. There is no way to single out one particular inertial frame, and hence absolute space is not necessary to account for the phenomenon of inertia. As Ernst Mach has pointed out, it cannot be a coincidence that the fixed stars appear indeed fixed relative to inertial frames, and hence that it is reasonable to consider inertia as a force exerted on local objects by the totality of the objects in the entire universe. Thus, Newton's law may best be interpreted as a consequence of the basic axiom that the sum of the forces, including the inertial force, acting on a particle should be zero and of the constitutive law of inertia, which states that this inertial force should be given by $-m\mathbf{a}$, where $\mathbf{a}$ is the acceleration relative to an *inertial* frame.

3. Pre-classical spacetime.

Newton also discussed the idea of absolute time. It is important to understand that there is no parallel between absolute time and absolute space. In classical physics, absolute time is not problematical, and neither Berkeley nor Mach had any quarrel with it.

So far, we have implicitly assumed that the underlying infrastructure of classical physics is what V. Matsko and I called **pre-classical spacetime** in the book [MN]. (Earlier, in 1967, I called it "neo-classical space-time" in [N8]). Pre-classical spacetime is a mathematical structure whose ingredients are the following:

1) A set called the **eventworld**,

2) a relation on the eventwold called **precedence**,

3) a function, called the **timelapse** function, which assigns to each pair of events, with the first preceding the second, the time lapse between them, and

4) a function, called the **distance** function, which assigns to each pair of simultaneous events the distance between them. (A pair of events is called simultaneous if each precedes the other or, equivalently, if the time lapse between them is zero).

A precise mathematical description, including appropriate axioms, is described in Chapter 4 of [MN]. The ingredients mentioned above all have an operational meaning: time lapses correspond to measurements with a stopwatch; distances correspond to measurements with a tape-measure, made at a particular instant.

Simultaneity is an equivalence relation on the event world and the corresponding equivalence classes are called **instants**. The set of all instants is what one might call "absolute time". The distance function endows each instant with the structure of a three-dimensional Euclidean space. Thus the instants are really instantaneous spaces. However, there is no natural Euclidean space that is independent of time. A **worldline** is a function that assigns to each instant an event that belongs to this instant. The distance between two worldlines at a given instant is simply the distance between the simultaneous events that the two worldlines assign to this instant. If this distance does not depend on the instant, we say that the two worldlines have *constant distance*. A frame of reference can now be defined to be a partition of the eventworld into worldlines, any two of which have constant distance. Such a frame of reference then has the natural structure of a Euclidean space. A location in such a frame is simply one of the worldlines the frame consists of. A detailed explanation of the pre-classical spacetime structure can be found in [MN].

4. Relativistic spacetime.

Under everyday circumstances, measurements of time lapses and distances are not problematical. They become problematical, however, when it matters that the transmission of information by light or other electromagnetic means is not instantaneous. It should be clear by now that it makes no sense to speak about the speed of light except relative to some frame of reference. Light being a wave phenomenon, this speed should be interpreted as the speed relative to the frame of reference defined by the medium that carries the wave, as is the speed of sound or the speed of water-waves. The physicists of the 19th century knew this, of course, so they invented the "luminiferous ether" as the carrier of light- and electromagnetic waves. The question that immediately arises is this: How does the earth and other astronomical objects move relative to the luminiferous ether. We all have heard of the expensive experiments of Michaelson and Morley that were designed to find out. In the end, it was Einstein who realized that there is no luminiferous ether and that entirely new and counterintuitive spacetime structures are needed to account for what happens in the real world in situations in which the transmission of information cannot

be regarded as instantaneous. For such spacetime structures, the precedence relation is no longer total, i.e. there may be pairs of distinct events neither of which precedes the other. Simulaneity can no longer be given a useful meaning and there is no absolute time. The "speed of light" is no longer the speed of anything, it becomes merely a unit conversion factor. There are no relativistic counterparts to frames of reference that are not inertial. A detailed explanation of the spacetime of Special Relativity can be found in [MN].

Note:

I sent a copy of an earlier version of this paper to Steven Pinker. Here is his answer by email, dated Oct 6, 1998:

"Dear Professor Noll,

Many thanks for your kind words, and for sending me your fascinating paper on the illusion of physical space. The central thesis is an interesting consequence of the psychology of space, and quite convincing.

With best wishes, Steve Pinker"

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[2] On Material Frame-Indifference

1. Introduction.

There is a considerable amount of confusion in the literature about the meaning of material frame-indifference, even among otherwise knowledgeable people. This paper is an attempt at clarification. I first started thinking about this issue when I was a student and tried to learn the theories of linearized elasticity and of linearly viscous fluids (a.k.a. Navier-Stokes fluids). In the former, one assumes that the stress $\mathbf{T}$ at a material point is determined by the gradient of the displacement-vector field $\mathbf{u}$. In the latter, one assumes that the stress $\mathbf{T}$ at a material point is determined by the gradient of the velocity field $\mathbf{v}$ and the density. For either case, denote the function which describes the dependence on the gradient by $\hat{\mathbf{T}}$. Then the following additional assumptions are most often introduced, not necessarily in the order and in the form given here.
(1) $\hat{\mathbf{T}}(\mathbf{A})$ depends only on the symmetric part of $\mathbf{A}$, i.e. *)

$$\hat{\mathbf{T}}(\mathbf{A}) = \hat{\mathbf{T}}(\frac{1}{2}(\mathbf{A} + \mathbf{A}^\top)) \quad \text{for all lineons } \mathbf{A}.$$

(2) The function $\hat{\mathbf{T}}$ is linear.
(3) We have $\hat{\mathbf{T}}(\mathbf{QEQ}^\top) = \hat{\mathbf{T}}(\mathbf{E})$ for all symmetric lineons $\mathbf{E}$ and all orthogonal lineons $\mathbf{Q}$.

Using some fairly elementary pure mathematics, it is then proved that the function $\hat{\mathbf{T}}$ must be given by the following specific formula:**)

$$\hat{\mathbf{T}}(\mathbf{E}) = 2\mu\mathbf{E} + (\lambda(\mathrm{tr}\mathbf{E}) + p)\mathbf{1}_{\mathcal{V}} \quad \text{for all symmetric lineons } \mathbf{E}.$$

(In the case of elasticity, μ is the shear modulus, $\lambda + \frac{2}{3}\mu$ is the modulus of compression, and p is most often assumed to be zero. In the case of viscous fluid theory μ is the shear viscosity, $\lambda + \frac{2}{3}\mu$ is the bulk viscosity, and p is the pressure, all of which may depend on the density.)

Most of the justifications that I found in the textbooks for these assumptions were mysterious to me. I was not satisfied by the justification that the two theories have been spectacularly successful for describing many physical phenomena and for designing machines, bridges, ships, airplanes, etc.. I now know that the assumption (1) is a consequence of the principle of frame-indifference. In the case of viscous fluid theory, assumption (3) also follows from

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*) We use the mathematical infrastructure, notation, and terminology of [FDS]. In particular, we use "lineon" as a contraction of "linear transformation from an linear space to itself". In [NLFT] the term "tensor" is used instead of "lineon". I pointed out in [N10] that "tensor" has a much more general meaning and lineon is just a special case. Given a lineon $\mathbf{A}$, we denote its transpose (a.k.a."adjoint") by $\mathbf{A}^\top$ and its trace by tr$\mathbf{A}$.

**) $\mathbf{1}_{\mathcal{V}}$ denotes the identity mapping of the underlying vector space $\mathcal{V}$.

the principle of frame-indifference. In the case of elasticity, however, assumption (3) means that the material possesses a special kind of material symmetry, namely isotropy; there are many elastic materials for which (3) is not appropriate. In the case of elasticity, assumption (2) follows from the fact that the theory is obtained by linearization from the theory of finite elasticity and can be valid only approximately for small deformations from a stress-free "natural" configuration. (If large deformations are taken into account, then assumption (1) becomes inconsistent with the principle of frame-indifference.) In the case of viscous fluid theory, the only justification for assumption (2) that I know of is that it comes out as a first approximation when one applies the retardation theorem for general simple fluids as described in [CN].

In classical particle physics it is often assumed that the force that is exerted at a given instant by one particle on another depends only on the position of the two particles. Then the following additional assumptions are introduced:

(a) The force has the direction given by the straight line joining the two particles.
(b) The magnitude of the force depends only on the distance between the two particles.

The textbooks do not usually give any convincing reasons for making these two assumptions beyond claiming that they are reasonable. In fact, both of them are consequences of the principle of material frame-indifference, as we will prove in Sect.4.

In recent years, the theory of liquid crystals has become a very popular subject. A large part of this theory deals only with bodies that are not subject to deformation, but only to changes involving a *director field* $\mathbf{n}$, which influences the optical and electromagnetic properties of the body. It is assumed that this director field gives rise to a free energy with a density σ per unit volume, and that this density at a given material point depends only on the values of $\mathbf{n}$ and its gradient at that point. Denote the function which describes this dependence by $\hat{\sigma}$. The following additional assumption is then introduced: The identity

$$\hat{\sigma}(\mathbf{Qu}, \mathbf{QGQ}^\top) = \hat{\sigma}(\mathbf{u}, \mathbf{G}) \quad \text{for all vectors } \mathbf{u} \text{ and all lineons } \mathbf{G}$$

is valid for all proper orthogonal lineons $\mathbf{Q}$ when dealing with cholesteric liquid crystals or for all orthogonal lineons $\mathbf{Q}$, proper or not, when dealing with nematic liquid crystals. Some authors have claimed that this assumption is justified by the principle of frame-indifference. (Even "Galilean Invariance" has been invoked by some as a justification.) These authors are mistaken. Rather, the assumption expresses a certain kind of material symmetry. Roughly, it states that the director-field interacts isotropically (or hemitropically in the cholesteric case) with the underlying body. In other words, the body has no implicit preferred directions in addition to the explicit one given by the director-field itself. It is quite conceivable that there are materials for which this assumption fails to be appropriate, although one should not call such materials "*liquid* crystals".

The term "principle of material frame-indifference" was introduced in 1965 by C. Truesdell and me in our contribution [NLFT] to the Encyclopedia of

Physics. Earlier, I had used the term "principle of objectivity", and some people use this term to this day. I meant "objectivity" to express independence of the "observer", but Truesdell disliked the term as being too easily misinterpreted. In fact, "observer" is also easily misinterpreted; a much better term is "frame of reference", or "frame" for short. In my doctoral thesis in 1954, I used the term "principle of isotropy of space", but I discarded it soon thereafter because I realized that there is really no such thing as (physical) "space". Of course, the principle has been applied implicitly for a long time without the use of an explicit name or formulation.

2. Frames of reference, motion.

As stated in the first paper [1] of this collection, a frame of reference should make it possible to speak of *locations*. Mathematically, locations are points in a (genuine) *Euclidean space* (as defined precisely in Chapter 4 of [FDS]), which we will then call a **frame-space**. Such a frame-space can be constructed from a suitable rigid material system (the most important example is the earth). Mathematically, such a rigid system is a metric set, i.e., a set $\mathcal{S}$ endowed with structure by the specification of a function $d : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{P}$ *). Given any two points x and y of the rigid system, $d(x, y)$ should be interpreted as the distance from x to y as measured, for example, with a measuring tape. The observed facts of such measurements show that $\mathcal{S}$ is isometric to a suitable subset of any given 3-dimensional Euclidean space. Of course, there are infinitely many such spaces, any two of which are isomorphic. In Sect.6 I will show how one can construct, by an intrinsic mathematical construction, a particular such space, and how the given rigid system can be imbedded in that space, which we then call the *frame-space determined by the given rigid system.*

As stated in [1] one can use geometric concepts such as *straight line, direction, angle*, etc. only when such a frame-space is at hand. Members of the *translation-space* of the frame-space are then called *vectors.* Hence concepts such as *velocity, acceleration, kinetic energy*, and *force* also require the specification of a frame-space.

If one deals only with the internal properties of a body not subject to deformation, one can use the body itself as a metric set from which a frame-space can be constructed. In this case it is not natural to consider any other frame-space, and hence frame-indifference is not an issue. This is the case for the theory of liquid crystals mentioned above. However, frame-indifference does come into play in theories that deal with *deforming* liquid crystals.

3. Inertia.

In elementary science and physics courses, students are very often confronted with statements such as "a particle will move along a straight line with

*) $\mathbb{P}$ denotes the set of all positive real numbers (including zero). We will use $\mathbb{R}$ to denote the set of *all* real numbers.

constant speed unless it is subject to an outside force." Later they will learn about "Newton's law" $\mathbf{f} = m\mathbf{a}$: the force acting on a particle is proportional to its acceleration, its inertial mass m being the proportionality factor. As was pointed out in the previous paper [1] of this collection, these statements aquire a meaning only after a frame of reference has been specified. They *cannot* be valid relative to *every* frame of reference. In fact, one can always construct frames relative to which the particle will undergo any motion prescribed at will. Newton's law is valid only in certain preferred frames, which are called **inertial frames**. There are infinitely many inertial frames, any one of which moves relative to any other in a uniform translational motion. Hence the laws of inertia remain valid under such changes of frame. This fact is often called "Galilean Invariance". It turns out that the frame of reference determined by the fixed stars and the condition that the center of the sun be at rest is, for most practical purposes, an inertial frame. This fact, together with the inverse-square law for gravitation, made it possible not only to explain the orbits of the planets and their moons, but even to make accurate predictions about the orbits of artificial satellites. Thus Newtonian mechanics became, perhaps, the first triumph of modern mathematical science. However, I believe that Newton's *absolute space* is a chimera.

Inertia plays a fundamental role in classical particle mechanics and also in the mechanics that deals with the motion of rigid bodies. However, when dealing with deformable bodies, inertia plays very often a secondary role. In some situations, it is even appropriate to neglect inertia altogether. For example, when analyzing the forces and deformations that occur when one squeezes toothpaste out of a tube, inertial forces are usually negligible. Thus, I believe that the basic concepts of mechanics in general should *not* include items such as momentum, kinetic energy, and angular momentum, because they are relevant only when inertia is important. What remains are the two fundamental balance laws:

(1) The sum of all the forces (including the inertial forces) acting on a system or any of its parts should be zero.
(2) The sum of the moments (including the moments of inertial forces) acting on a system or any of its parts should be zero.

As far as these balance laws are concerned, inertial forces should be treated on equal terms with other kinds of forces. In this context, Newton's law $\mathbf{f} = m\mathbf{a}$ should be viewed as the result of the combination of two laws. The first is the force-balance law in the form $\mathbf{f} + \mathbf{i} = \mathbf{0}$, where $\mathbf{f}$ denotes the sum of the non-inertial forces acting on the particle, while $\mathbf{i}$ denotes the inertial force acting on it. The second law is the constitutive law of inertia. It states that $\mathbf{i} = -m\mathbf{a}$ when an inertial frame of reference is used. If the frame used is arbitrary, not necessarily inertial, the constitutive law of inertia takes the form

$$\mathbf{i} = -m(\mathbf{u}^{\bullet\bullet} + 2\mathbf{A}\mathbf{u}^{\bullet} + (\mathbf{A}^{\bullet} - \mathbf{A}^2)\mathbf{u}). \tag{1}$$

Here, the value $\mathbf{u}(t)$ of the function $\mathbf{u}$ at time t denotes the position vector of the particle relative to a reference point (often called "origin") which is at

rest in some inertial frame, although not necessarily in the frame used. The value $\mathbf{A}(t)$ of the function $\mathbf{A}$ at time t is a skew lineon; it measures the rate of rotation of the given frame relative to some inertial frame. Dots denote time-derivatives. If the refence point is at rest not only in some inertial frame but also in the frame used and if $\mathbf{A}$ is constant, i.e. if $\mathbf{A}^\bullet$ is zero, then the contributions to the inertial force given by the second and third term on the right of (1) are called *Coriolis* force and *centrifugal* force, respectively. For example, the frame of refence determined by the earth is approximately inertial for small-scale phenomena, but the contribution of the Coriolis force can be decisive when large-scale wind or ocean-current phenomena are analyzed. In his famous pendulum experiment in 1851, Foucault demomstrated that the Coriolis force can be detected even on a small scale. This effect is nowadays used in gyrocompasses.

4. Frame-Indifference.

Physical processes are usually described in a mathematical framework provided by a frame of refence with its corresponding frame-space $\mathcal{F}$. Concepts such as *vector*, *lineon*, or *tensor* become meaningful only relative to the given frame-space. For example, a vector is a member of the translation-space $\mathcal{V}$ of $\mathcal{F}$, and a lineon is a member of the space $\mathrm{Lin}\,\mathcal{V}$ of all linear transformations of $\mathcal{V}$ into itself.

Now consider two frames of refence with corresponding frame-spaces $\mathcal{F}$ and $\mathcal{F}'$ and denote their translation spaces by $\mathcal{V}$ and $\mathcal{V}'$, repectively. If x is the position of a material point or particle at a given time t in the frame-space $\mathcal{F}$, then the position of the *same* particle at the *same* time t in the frame-space $\mathcal{F}'$ will be given by $x' = \alpha_t(x)$ where $\alpha_t : \mathcal{F} \longrightarrow \mathcal{F}'$ is an isometry and hence a Euclidean isomorphism (see Sect.45 of [FDS]). The mapping $t \mapsto \alpha_t$ describes the motion of the frame $\mathcal{F}$ as seen in the frame $\mathcal{F}'$. The gradient $\mathbf{A}_t := \nabla\alpha_t$ (see Sect.33 of [FDS]) is an inner-product preserving linear mapping from $\mathcal{V}$ onto $\mathcal{V}'$. A vector $\mathbf{u}$ and a lineon $\mathbf{L}$ relative to the frame-space $\mathcal{F}$ at time t will appear as $\mathbf{u}' = \mathbf{A}_t\mathbf{u}$ and $\mathbf{L}' = \mathbf{A}_t\mathbf{L}\mathbf{A}_t{}^\top$, repectively, relative to the frame-space $\mathcal{F}'$. To say that descriptions in two frame spaces $\mathcal{F}$ and $\mathcal{F}'$ describe the same physical process means that these descriptions must be isomorphic. Therefore, there must be a fixed Euclidean isomorphism $\beta : \mathcal{F}' \longrightarrow \mathcal{F}$ with the following property: If we replace the isomorphism α_t above with the composite $\gamma_t := \beta \circ \alpha_t$, which is a Euclidean automorphism of $\mathcal{F}$, then the fomulas for a change of frame given above remain valid with $\mathcal{F}'$ and $\mathcal{V}'$ replaced by $\mathcal{F}$ and $\mathcal{V}$, repectively, i.e. we have $\mathbf{u}' = \mathbf{Q}_t\mathbf{u}$ and $\mathbf{L}' = \mathbf{Q}_t\mathbf{L}\mathbf{Q}_t{}^\top$, where $\mathbf{Q}_t := \nabla\gamma_t = \nabla\beta \circ \mathbf{A}_t$ is an inner-product automorphism of $\mathcal{V}$, i.e. an orthogonal lineon $\mathbf{Q}_t \in \mathrm{Orth}\mathcal{V}$.

Consider now a physical system. The **principle of material frame-indifference**, as applied to this system, can then be formulated as follows:

The constitutive laws governing the internal interactions between the parts of the system should not depend on whatever external frame of reference is used to describe them.

The principle should apply to interactions of any kind, be they mechanical, thermodynamical, optical, electromagnetic, or whatever.

It is important to note that the principle applies only to *internal* interactions, not to actions of the environment on the system and its parts, because usually the frame of reference employed is actively connected with the environment. For example, if one considers the motion of a fluid in a rigid container, one usually uses the frame of reference determined by the container, which certainly affects the fluid. Inertia should always be considered as an action of the environment on the given system and its parts, and hence its description does depend on the frame of reference used. As we saw in Sect.3, the inertial force has a simple description only when an inertial frame is used.

In many cases, gravitational, electric, and magnetic fields are generated by entities that are outside the system under consideration. Hence the laws that describe the response of the parts of the system to these fields are not frame-indifferent. Perhaps one should call such laws "external constitutive equations". These laws may have certain symmetry properties, but these should not be confused with frame-indifference. For example, the most common external constitutive law is the law $\mathbf{i} = -m\mathbf{a}$ for the inertial force. It is valid for all inertial frames. Its symmetry, usually called "Galilean invariance", has nothing whatever to do with frame-indifference. Of course, laws that describe gravitational, electric, or magnetic interactions between parts of the given system are subject to the principle of frame-indifference. For example, the law of gravitational attraction between two particles, which has the form (12) stated below with $h(r) := m_1 m_2/r^3$ for all $r \in \mathbb{P}$, is frame-indifferent indeed.

It is also important to note that the principle applies only to *external* frames of reference, not to frames that are constructed from the system itself, as is the case in the theory of liquid crystals not subject to deformation described in Sect.1.

To illustrate how the principle of material frame-indifference is applied, we consider the simple example already mentioned in Sect.1, namely a system consisting of only two particles and the force-interaction between them.*) We assume that the force $\mathbf{f}$ exerted at a given time on the first particle by the second depends only on the positions of the two particles at that time. Denote the function that describes this dependence by $\hat{\mathbf{f}} : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{V}$, so that

$$\mathbf{f} = \hat{\mathbf{f}}(x, y) \tag{2}$$

is this force when the particles are located at x, and y, respectively. Now, after a change of frame given by the Euclidean automorphism $\gamma : \mathcal{F} \longrightarrow \mathcal{F}$ (the dependence of γ on time is not relevant in this particular case), the particles appear at the locations $x' = \gamma(x)$ and $y' = \gamma(y)$ and the force appears to be

*) In the derivation that follows, we use the concepts, notations, and results of Sects.32, 33, and of Chapt.4 of [FDS]. Essentially the same derivation was presented first in 1957 on pp.38-40 of [N9].

$\mathbf{f}' = \mathbf{Q}\mathbf{f}$, where $\mathbf{Q} := \nabla\gamma$. The principle of material frame-indifference states that the function $\hat{\mathbf{f}}$ should also describe the dependence of the force on the locations after the change of frame, so that

$$\mathbf{Q}\mathbf{f} = \mathbf{f}' = \hat{\mathbf{f}}(x', y') = \hat{\mathbf{f}}(\gamma(x), \gamma(y)). \tag{3}$$

Combining (2) and (3), we find that the function $\hat{\mathbf{f}}$ must satisfy

$$\mathbf{Q}\hat{\mathbf{f}}(x, y) = \hat{\mathbf{f}}(\gamma(x), \gamma(y)) \quad \text{when} \quad \mathbf{Q} := \nabla\gamma. \tag{4}$$

This equation should be valid for every Euclidean automorphisms γ of $\mathcal{F}$ and all points $x, y \in \mathcal{F}$. Now choose a point $q \in \mathcal{F}$ arbitrarily and define $\hat{\mathbf{g}} : \mathcal{V} \longrightarrow \mathcal{V}$ by

$$\hat{\mathbf{g}}(\mathbf{u}) := \hat{\mathbf{f}}(q, q + \mathbf{u}) \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}. \tag{5}$$

Let $x, y \in \mathcal{F}$ be given. We apply (4) to the case when γ is the translation $\mathbf{u} := q - x$ that carries x to q. Since the gradient of a translation is the identity and since $\mathbf{u}(x) = x + (q - x) = q$ and $\mathbf{u}(y) = y + (q - x) = q + (y - x)$, the equation (4) reduces to

$$\hat{\mathbf{f}}(x, y) = \hat{\mathbf{f}}(q, q + (y - x)) = \hat{\mathbf{g}}(y - x), \tag{6}$$

valid for all $x, y \in \mathcal{F}$. Recalling that the gradient $\mathbf{Q}$ of a given Euclidean automorphism γ is characterized by the condition that $\gamma(x) - \gamma(y) = \mathbf{Q}(x - y)$ holds for all $x, y \in \mathcal{F}$, we conclude from (6) and (4) that

$$\mathbf{Q}\hat{\mathbf{g}}(\mathbf{u}) = \hat{\mathbf{g}}(\mathbf{Q}\mathbf{u}) \tag{7}$$

must be valid for all $\mathbf{u} \in \mathcal{V}$ and all orthogonal lineons $\mathbf{Q}$. Given $\mathbf{u} \in \mathcal{V}$, the equation (7) must be valid, in particular, for all orthogonal $\mathbf{Q}$ that leave $\mathbf{u}$ unchanged. Hence $\hat{\mathbf{g}}(\mathbf{u})$ must also remain unchanged by these $\mathbf{Q}$. Since the only vectors that have this property are scalar multiples of $\mathbf{u}$, we conclude that there is a function $g : \mathcal{V} \longrightarrow \mathbb{R}$ such that

$$\hat{\mathbf{g}}(\mathbf{u}) = g(\mathbf{u})\mathbf{u} \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}. \tag{8}$$

In view of (7), this function must have the property that

$$g(\mathbf{u}) = g(\mathbf{Q}\mathbf{u}) \quad \text{for all orthogonal lineons} \quad \mathbf{Q}. \tag{9}$$

Now choose a unit vector $\mathbf{e}$ arbitrarily and define the function $h : \mathbb{P} \longrightarrow \mathbb{R}$ by

$$h(d) := g(d\mathbf{e}) \quad \text{for all} \quad d \in \mathbb{P}\,. \tag{10}$$

Given $\mathbf{u} \in \mathcal{V}$, it is easily seen that we can choose an orthogonal $\mathbf{Q}$ such that $\mathbf{u} = |\mathbf{u}|\mathbf{Q}\mathbf{e} = \mathbf{Q}|\mathbf{u}|\mathbf{e}$. Using (9) with $\mathbf{u}$ replaced by $|\mathbf{u}|\mathbf{e}$, it follows that

$$g(\mathbf{u}) = h(|\mathbf{u}|) \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}. \tag{11}$$

Combining (11) with (8) and (6), we see that the equation (2) for the dependence of the force $\mathbf{f}$ on the locations x and y must reduce to the specific form

$$\mathbf{f} = \hat{\mathbf{f}}(x, y) = h(|x - y|)(x - y), \tag{12}$$

which justifies the assumptions (a) and (b) stated in Sect.1.

For the derivation of (12) above, it was irrelevant that the Euclidean automorphisms γ for which (4) holds may depend on time. However, such possible time-dependence plays a crucial role, for example, when applying the principle of frame-indifference to derive the specific form of the constitutive equation for linearly viscous fluids discussed in Sect.1.

It is possible to make the principle of material frame-indifference vacuously satisfied by using an intrinsic mathematical frame-work that does not use a frame-space at all when describing the internal interactions of a physical system. I did this in 1972 in [N7] for the continuum mechanics of simple materials, defined in the technical sense of Sect.4 of the paper [3], which is the next in this collection. However, the mathematics that is needed to do this, although not necessarily complicated, is not familiar to many people and hence resisted by some as being "too abstract". Also, it seems that the action of the environment on a system cannot be described without using a frame of reference, and hence one must introduce such a frame in the end when dealing with specific problems.

5. Relativity.

Up to now, we have tacitly assumed the validity of the common-sense notions of time and distance. Specifically, we implicitly have taken for granted that the following statements are unambiguously valid:
(1) Any two given events are either simultaneous or one of them precedes the other.
(2) To any given two events one can assign a time-lapse, which is zero if and only if they are simultaneous.
(3) To any two simultaneous events one can assign a distance between them.
A precise mathematical structure that describes a world in which these assumptions are valid is described in Sect.4.1 of [MN] under the name of *Pre-classical Spacetime*. In the present paper we have given an intuitive idea of what is meant by a frame of reference; in Sect.4.2 of [MN] one can find a more precise definition in the context of pre-classical spacetime.

In the Theory of Relativity (both Special and General) the common-sense notions of time and distance can no longer be used: Simultaneity becomes meaningless. There can be two events neither of which precedes the other. Even if one event does precede another, the timelapse between them may depend on a world-path connecting them. There is no simple notion of distance, and hence one cannot define unambiguously what is meant by a rigid system. Therefore, there is no relativistic counterpart of a frame of reference. The only correlation one can make is between inertial frames and world-directions (as defined in Sect.53 of [MN]).

In view of these remarks, it is not clear what a relativistic counterpart to the principle of material frame-indifference would be. The only proposal for such a counterpart that I know of is the *principle of non-sentient response* of Bragg [B]. The idea behind this principle is the following : The dependence of the state of a given material point X on the world-paths of the material points surrounding X can only involve information about these world-paths that can be obtained by signals originating from the worldpath of X and reflected back from these nearby world-paths.

6. Construction of frame-spaces.

In this section, I will describe the construction of a Euclidean space from a given metric set, as already announced in Sect.2.

Assume that a metric set $\mathcal{S}$, endowed with structure by the prescription of a function $d : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{P}$, is given. We also assume that $\mathcal{S}$ is isometric to some subset of some Euclidean space. This means that we can choose a Euclidean space $\mathcal{E}$ and a mapping $\kappa : \mathcal{S} \longrightarrow \mathcal{E}$ such that

$$d(x, y) = |\kappa(x) - \kappa(y)| \quad \text{for all} \quad x, y \in \mathcal{S}. \tag{13}$$

We may assume, without loss, that $\mathcal{E}$ is the flat span of the range $\mathrm{Rng}\kappa$ of κ, i.e. the smallest subspace of $\mathcal{E}$ that includes $\mathrm{Rng}\kappa$, because otherwise we could replace $\mathcal{E}$ by this flat span.

We now define the mapping $\Phi : \mathcal{E} \longrightarrow \mathrm{Map}(\mathcal{S}, \mathbb{P})$ *) by

$$\Phi(z)(x) := |z - \kappa(x)| \quad \text{for all } z \in \mathcal{E},\ x \in \mathcal{S}. \tag{14}$$

Using Prop.7 of Sect.45 of [FDS] one can easily prove that the mapping Φ thus defined is injective and that $z \in \mathrm{Rng}\kappa$ if and only if $0 \in \mathrm{Rng}(\Phi(z))$. We now put

$$\mathcal{F} := \mathrm{Rng}\Phi \ \subset \ \mathrm{Map}(\mathcal{S}, \mathbb{P}) \tag{15}$$

and endow $\mathcal{F}$ with the structure of a Euclidean space by requiring that the invertible mapping $\Phi|^{\mathcal{F}} : \mathcal{E} \longrightarrow \mathcal{F}$ **) be a Euclidean isomorphism. Using Prop.5 of Sect.45 of [FDS] one can easily prove that the subset $\mathcal{F}$ of $\mathrm{Map}(\mathcal{S}, \mathbb{P})$, its structure as a Euclidean space, and the injective mapping

$$\phi := \Phi|^{\mathcal{F}} \circ \kappa : \ \mathcal{S} \longrightarrow \mathcal{F} \tag{16}$$

are all independent of the initial choice of $\mathcal{E}$ and κ. We call $\mathcal{F}$ the **frame-space** of the given metric set $\mathcal{S}$.

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*) Given any two sets S and T, $\mathrm{Map}(S, T)$ denotes the set of all mappings from S to T.

**) $\Phi|^{\mathcal{F}}$ is the mapping obtained from Φ by adjusting the codomain from its original, namely $\mathrm{Map}(\mathcal{S}, \mathbb{P})$, to the range $\mathcal{F}$.

We use the mapping (16) to imbed the metric set $\mathcal{S}$ into the frame-space $\mathcal{F}$ constructed from it. Then a given point $x \in \mathcal{S}$ becomes identified with the function $\phi(x) = d(x, \cdot)$, which gives the distances from x to all points in $\mathcal{S}$. Every point in the frame space $\mathcal{F}$ is identified with the function that gives its distances from all the points in $\mathcal{S}$.

The construction just given implies that locations in the frame-space constructed from a given rigid system $\mathcal{S}$ can be determined by distance measurements alone. In practice, however, it is often convenient to also use angle-measurements involving lines of sight.

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[3] Updating *The Non-Linear Field Theories of Mechanics*

0. Introduction

Clifford Truesdell was my thesis advisor when I was a graduate student at Indiana University in 1954. Later he invited me to become a co-author in a treatise which he initiated: *The Non-Linear Field Theories of Mechanics* ([NLFT]). I described the genesis of [NLFT] in detail in [N6]. Over the years [NLFT] became a basic reference treatise for continuum mechanics. It was reprinted in 1992, and a translation into Chinese appeared in 2002. A third edition, with a preface by Stuart Antman, appeared in 2004. Starting with my thesis, I introduced a mathematical infrastructure, mainly based on coordinate-free linear algebra, into continuum mechanics. I believed that this structure would provide better clarity and conceptual insight. Unfortunately, this infrastructure was familiar to almost nobody working in the field of continuum mechanics and was scornfully dismissed by some as lacking any value. Clifford Truesdell, on the other hand, had an open mind and soon began to see the value of my approach. Thus, [NLFT], which appeared in 1965, was largely written using this new mathematical infrastructure. In the preface to the third edition Stuart Antman wrote "It is interesting to note that Truesdell and Noll's system of notation, symbols, and terminology has been widely adopted, even by the scientific descendents of critics of their enterprise".

As I pointed out in the Introducton [N0] to this collection, [NLFT] is in many respects obsolete and perhaps should be updated after almost 40 years of its original publication. I believe that such an update should be very different from the original, and I would like to present here some guidelines. These guidelines call for a mathematical infrastructure that is more sophisticated than the one used in [NLFT]. It is outlined in Sect.2 below. I am under no illusion that it will be widely accepted easily and soon. Perhaps I am really ahead of my time, as the reviewer mentioned in the Introduction [N0] conjectured.

Guidelines for an update of [NLFT]

1. The role of mass

Gravitational attractions between parts of a body are negligeable except when dealing with very large bodies such a as whole planets or stars. Thus, inertial and gravitational mass should *not* occur in internal constitutive laws because they involve only external actions (namely inertia and gravitation) on the parts of a body. Contrary to the statement in Sect.15 of [NLFT], mass should *not* be assigned a priori as part of the specification of a body. In some situations, it is perfectly all right to neglect inertia or gravity or both. Thus, what is called *Cauchy's law af motion* (16.6) in [NLFT] should be replaced by

$$\mathrm{div}\mathbf{T} + \mathbf{b} = \mathbf{0}, \qquad \mathbf{b} = \mathbf{b}_{\mathrm{ni}} + \mathbf{i}, \tag{1.1}$$

where $\mathbf{b}$ is the *total* body force per unit volume, $\mathbf{b}_{\rm ni}$ is the *non-inertial* body force per unit volume, and $\mathbf{i}$ is the *inertial* force per unit volume. The latter is given by $\mathbf{i} = -\rho\mathbf{a}$, where ρ is the mass density and $\mathbf{a}$ the acceleration, *only* when an inertial frame of reference is used. When the frame is not inertial, Cauchy's law (1.1) is still valid, but one must use the more complicated formula for $\mathbf{i}$ obtained from (1) on p.15 of [N2] by replacing m with ρ. It is more appropriate to call (1.1) *Cauchy's balance law* because it is not necessarily related to any motion.

Remark 1: In classical "analytical" mechanics, which deals only with point-particles and rigid bodies, inertia does play an essential role and can never be neglected. (See the beginning of the Introduction to [NLFT].) Continuum mechanics is conceptually quite different, and inertia should *not* be part of its basic principles. This insight came to me only after the first edition of [NLFT] was published in 1965. ∎

2. Mathematical infrastructure.

A good update of [NLFT] should be based on a much more effective mathematical infrastructure. Better familiarity is needed with **linear spaces** that are not only *not* $\mathbb{R}^n$, but *not* even innner-product spaces. Also, one should be comfortable with constructions *) of new linear spaces from one or more given ones. For example, given a linear space $\mathcal{T}$, one can construct $\mathcal{T}^*$, the dual of $\mathcal{T}$, the space $\mathrm{Lin}\,\mathcal{T}$ of all lineons (linear transformations)**) on $\mathcal{T}$, and the space $\mathrm{Lin}\,(\mathcal{T},\mathcal{T}^*) \cong \mathrm{Lin}_2(\mathcal{T}\times\mathcal{T},\mathbb{R})$ of all linear mappings from $\mathcal{T}$ to its dual $\mathcal{T}^*$, which is naturally identified with the space of all bi-linear forms on $\mathcal{T}$. The symmetric members of this space form a subspace $\mathrm{Sym}(\mathcal{T},\mathcal{T}^*)$ and the skew-symmetric ones a subspace $\mathrm{Skew}(\mathcal{T},\mathcal{T}^*)$. The set of positive (a.k.a. positive semi-definite) members of $\mathrm{Sym}(\mathcal{T},\mathcal{T}^*)$ form a linear cone $\mathrm{Pos}(\mathcal{T},\mathcal{T}^*)$, and its strictly positive (a.k.a. positive-definite) members a linear cone $\mathrm{Pos}^+(\mathcal{T},\mathcal{T}^*)$. Spaces such as $\mathrm{Lin}\,(\mathrm{Lin}\,\mathcal{T})$ are needed, for example, in the theory of elasticity. (To call them "4th order tensors" only obscures the true nature of the concepts in question.) To endow the given space $\mathcal{T}$ with the structure of an **inner-product space**, one singles out a particular member $\mathrm{ip} \in \mathrm{Pos}^+(\mathcal{T},\mathcal{T}^*)$ and uses it to identify $\mathcal{T}^*$ with $\mathcal{T}$.

Of great importance, also, is an intrinsic differential calculus involving finite dimensional spaces. A fairly full treatment of these matters is presented in my book [FDS] of 1987, which is now also available on my website

One should also be aware that a fundamental part of mathematics is the study of **mathematical structures**, which are defined by specifying

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*) I call such constructions "tensor functors". A detailed description is given in [N10].

**) In [NLFT] the term "tensor"is used instead of "lineon". I pointed out in [N10] that "tensor" has a much more general meaning and lineon is just a special case. I introduced the contraction "lineon" for "linear transformation" in [FDS].

ingredients and **axioms**. For example, the ingredients for a structure of a *continuous body system* as given in Sect.3 below are the set $\mathcal{B}$ and the class $\mathrm{Pl}\,\mathcal{B}$; the axioms are the requirements (B_1), (B_2), and (B_3). The ingredients of the (fairly complicated) structure of a *state space system* as described in Sect.7 are the sets $\mathcal{C}, \Pi, \Sigma, \mathcal{R}$, the mappings $\hat{\mathbf{C}}, \hat{\mathbf{R}}$, and the functional $\hat{\rho}$. There are five axioms, stated in Sect.7.

For each species of mathematical structures, one has a natural notion of **isomorphism**. The set of all **automorphism** of a particular structure, i.e the isomorphism from that structure to itself, give the **symmetry group** of that structure.

3. Body systems, placements, configurations, motions, and deformations.

In Sect.1 of [NLFT] a *body* is simply defined to be a three-dimensional differentiable manifold. First, the term *body* should be replaced by *continuous body system*, because one would not require it to have the connectedness implied by the term *body* alone.

To give a modern precise definition one first has to introduce two classes:

(i) A class Fr consisting of subsets of three-dimensonal Euclidean spaces that are candidates for regions occupied by a body system when placed in a frame of reference,

(ii) a class Tp of mappings which are candidates for changes of placement of a body system from one frame to another or itself.

We call the members of Tp *transplacements*. It serves well to take Fr to be the class of all *fit regions* introduced in [NV]. The set of fit regions included in a given Euclidean space $\mathcal{F}$ will be denoted by $\mathrm{Fr}\,\mathcal{F}$; its members are all open subsets of $\mathcal{F}$. We take Tp to be the class determined by the following requirements:

(T_1) Every $\lambda \in \mathrm{Tp}$ is an invertible mapping whose domain $\mathrm{Dom}\,\lambda$ and whose range $\mathrm{Rng}\,\lambda$ are subsets of Euclidean spaces $\mathrm{Dsp}\lambda$ and $\mathrm{Rsp}\lambda$, which are called the **domain-space** and **range-space** of λ, respectively.

(T_2) For every $\lambda \in \mathrm{Tp}$, there is a C^2- diffeomorphism *) $\phi : \mathrm{Dsp}\lambda \longrightarrow \mathrm{Rsp}\lambda$ such that $\lambda = \phi|^{\mathrm{Rng}\,\lambda}_{\mathrm{Dom}\,\lambda}$.

Definition. *A* **continuous body system** $\mathcal{B}$ *is a non-empty set endowed with structure by the specification of a non-empty class* $\mathrm{Pl}\,\mathcal{B}$ *satisfying the following requirements:*

(B_1) Every $\kappa \in \mathrm{Pl}\,\mathcal{B}$ is an invertible mapping with $\mathrm{Dom}\,\kappa = \mathcal{B}$ and $\mathrm{Rng}\,\kappa \in \mathrm{Fr}$.

(B_2) For all $\kappa, \gamma \in \mathrm{Pl}\,\mathcal{B}$ we have $\kappa \circ \gamma^{\leftarrow} \in \mathrm{Tp}$.

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*) In most contexts, C^2 is good enough, but there may be situations where C^2 should be replaced by C^n with $n > 2$.

(B$_3$) For every $\kappa \in \mathrm{Pl}\,\mathcal{B}$ and $\lambda \in \mathrm{Tp}$ such that $\mathrm{Rng}\,\kappa = \mathrm{Dom}\,\lambda$ we have $\lambda \circ \kappa \in \mathrm{Pl}\,\mathcal{B}$.

The elements of $\mathcal{B}$ are called **material points**. The members of $\mathrm{Pl}\,\mathcal{B}$ are called **placements of** $\mathcal{B}$. Given $\kappa \in \mathrm{Pl}\,\mathcal{B}$, we call $\mathrm{Rng}\,\kappa$ the *region occupied by* $\mathcal{B}$ in the placement κ. The Euclidean space in which $\mathrm{Rng}\,\kappa$ is a fit region is denoted by $\mathrm{Frm}\,\kappa$ and is called the **frame-space** of κ. Its translation space is a three-dimensional inner-product space denoted by $\mathrm{Vfr}\,\kappa$ and is called the **frame-vector-space** of κ.

It is clear that a body system has the structure of a C^2-manifold, and hence also the strucure of a topological space, but the axioms above imply more than just that. More details can be found in [N10]. If a continuous body system is connected, we call it simply a **body**.

Given a body system $\mathcal{B}$ and a placement $\kappa \in \mathrm{Pl}\,\mathcal{B}$, we define $d_\kappa : \mathcal{B}\times\mathcal{B} \longrightarrow \mathbb{P}$ by

$$d_\kappa(X,Y) = |\kappa(X) - \kappa(Y)| \quad \text{for all} \quad X, Y \in \mathcal{B}. \tag{3.1}$$

Then d_κ is a metric on the set $\mathcal{B}$ that is isometric to the subset $\mathrm{Rng}\,\kappa$ of the Euclidean space $\mathrm{Frm}\,\kappa$. We call this metric the **configuration** induced by the placement κ. It is clear that one and the same configuration can be induced by infinitely many placements. (See also Sect.6 of the previous paper [N2].)

We now assume that a continuous body system $\mathcal{B}$ is given. When describing the behavior of the body in some environment, one must usually use a *frame of reference*, which is described mathematically by a three-dimensional Euclidean space $\mathcal{F}$, called the *frame space*. The translation space $\mathcal{V}$ of $\mathcal{F}$ is a 3-dimensional inner product space.

A **motion** of $\mathcal{B}$ in $\mathcal{F}$ is a mapping $\mu : \mathcal{B} \times I \longrightarrow \mathcal{F}$, where I is a genuine closed interval, with the following property: For every $t \in I$ the mapping $\mu_t : \mathcal{B} \longrightarrow \mathcal{R}_t := \mathrm{Rng}\,\mu(\cdot, t)$ defined by

$$\mu_t(X) := \mu(X,t) \qquad \text{for all} \qquad X \in \mathcal{B} \tag{3.2}$$

is a *placement* of $\mathcal{B}$ in $\mathcal{F}$ (in the sense of the precise definition given above). We usually assume that the left endpoint of I is 0. Let a motion μ be given. Then $d_\mu : \mathcal{B} \times \mathcal{B} \times I \longrightarrow \mathbb{P}$, defined by

$$d_\mu(X,Y,t) := d_{\mu_t}(X,Y) \qquad \text{for all} \quad t \in I, \quad X, Y \in \mathcal{B} \tag{3.3}$$

is called the **deformation process** induced by μ. If the motion is a rotation or translation, then the induced deformation process is just a freeze at a fixed configuration.

4. Body elements, simple materials.

Let a continuous body system $\mathcal{B}$ and material point $X \in \mathcal{B}$, as defined in Sect.3, be given. Since $\mathcal{B}$ has the structure of a C^2-manifold, we can consider the tangent space $\mathcal{T}_X$ and call it the **body element** of $\mathcal{B}$ at X, because it is the

precise mathematical realization of what is often represented as an "infinitesimal element" of the body. It is very important to be aware that $\mathcal{T}_X$ is just a three-dimensional linear space and not an inner product space.

Let a placement κ of $\mathcal{B}$ be given. We use $\mathcal{F} := \mathrm{Frm}\,\kappa$ and $\mathcal{V} := \mathrm{Vfr}\,\kappa$ for the frame-space and frame-vector-space of $\mathcal{B}$, respectively. Since κ is an invertible mapping of class C^2 from $\mathcal{B}$ to an open subset of $\mathcal{F}$, we can consider its gradient at X:

$$\mathbf{K} := \nabla_X \kappa \ \in \ \mathrm{Lis}\,(\mathcal{T}_X, \mathcal{V}). \tag{4.1}$$

We call $\mathbf{K}$ the **placement** of the body element $\mathcal{T}_X$ in the frame-vector space $\mathcal{V}$ induced by the placement κ. Since $\mathcal{V}$ is an inner-product space, it can be identified with its dual and hence the transpose $\mathbf{K}^\top$ is a member of the space $\mathrm{Lis}\,(\mathcal{V}, \mathcal{T}_X^*)$. Therefore, we can consider

$$\mathbf{G} := \mathbf{K}^\top \mathbf{K} \ \in \mathrm{Pos}^+(\mathcal{T}_X, \mathcal{T}_X^*). \tag{4.2}$$

Now, $\mathbf{G}$ depends on the placement κ only through the configuration d_κ defined by (3.1). Hence we call $\mathbf{G}$ the **configuration** of the element $\mathcal{T}_X$ induced by the configuration d_κ. This d_κ is a metric that induces a Riemannion structure on the body-manifold. This structure assigns a "metric tensor" to each point X in the body system. The configuration $\mathbf{G}$ is nothing but this metric tensor.

Now let a motion $\mu : \mathcal{B} \times I \longrightarrow \mathcal{F}$ of $\mathcal{B}$ in a given frame-space $\mathcal{F}$, as defined in the previous section, be given. We call the mapping $\mathbf{M} : I \longrightarrow \mathrm{Lis}\,(\mathcal{T}_X, \mathcal{V})$ defined by

$$\mathbf{M}(t) := \nabla_X \mu_t \qquad \text{for all} \qquad t \in I \tag{4.3}$$

the **motion of the element** $\mathcal{T}_X$ induced by the motion μ of the whole body system $\mathcal{B}$. Of course, the value $\mathbf{M}(t)$ at a given $t \in I$ is simply the placement of $\mathcal{T}_X$ induced by the placement μ_t of $\mathcal{B}$.

The mapping $\mathbf{P} : I \longrightarrow \mathrm{Pos}^+(\mathcal{T}_X, {\mathcal{T}_X}^*)$, defined by

$$\mathbf{P}(t) := \mathbf{M}(t)^\top \mathbf{M}(t) \quad \text{for all} \ \ t \in I \tag{4.4}$$

is then called the **deformation process** of the element $\mathcal{T}_X$ induced by the deformation process d_μ of the whole body system $\mathcal{B}$ as given by (3.3).

We assume now that the mapping $\mathbf{M}$ defined by (4.3) is differentiable. Its derivative is then given by

$$\mathbf{M}^\bullet = \mathbf{L}\mathbf{M}, \tag{4.5}$$

where the value of $\mathbf{L} : I \longrightarrow \mathrm{Lin}\,(\mathcal{V})$ at time t gives the velocity gradient of the body at the location of the material point at that time. Recall that the values of the symmetric and skew parts of $\mathbf{L}$, i. e.

$$\mathbf{D} := \frac{1}{2}(\mathbf{L} + \mathbf{L}^\top)\ , \quad \mathbf{W} := \frac{1}{2}(\mathbf{L} - \mathbf{L}^\top)\ , \ \ \text{with} \ \ \mathbf{L} = \mathbf{D} + \mathbf{W}\ , \tag{4.6}$$

give the corresponding *stretching* and *spin*, respectively. (See, for example, equation (25.6) in [NLFT].) Differentiating (4.4) and using (4.5) and (4.6) we easily obtain

$$\mathbf{P}^{\bullet} = 2\mathbf{M}^{\top}\mathbf{D}\mathbf{M}. \qquad (4.7),$$

relating the **rate of deformation** $\mathbf{P}^{\bullet}$ to the stretching $\mathbf{D}$. If $\mathbf{P}$ is n times differentiable, we have

$$\mathbf{P}^{(n)} = \mathbf{M}^{\top}\mathbf{A}_n\mathbf{M}. \qquad (4.8),$$

relating the **n'th rate of deformation** $\mathbf{P}^{(n)}$ to the *n'th Rivlin-Ericksen tensor* $\mathbf{A}_n$, as defined by (24.14) in [NLFT]. Thus, these rates of deformation are simply frame-free counterparts of the Rivlin-Ericksen tensors.

Material properties of body systems are local. These material properties may change from material point to material point. Generalizing what is called the "Principle of local action" in Sect.26 of [NLFT], one should formulate this postulate as follows:

The constitutive laws that describe the material properties at a material point X in a continuous body system $\mathcal{B}$ should involve only arbitrarily small neighborhoods of X in $\mathcal{B}$.

We say that $\mathcal{B}$ consists of a **simple material** if these constitutive laws for every material point X involve only the body element $\mathcal{T}_X$. A precise way of formulating such laws is to put them in the form of a mathematical structure, as in the example presented in Sect.7 below. We say that the body system is **materially uniform** if the structures for any two material points are isomorphic. One must observe the distinction between material uniformity and homogeneity as described in Sect.27 of [NLFT].

5. Reference-free constitutive laws.

In the past, the formulation of constitutive laws often involved the use of a reference placement. Here is a passage from Sect.21 of [NLFT]:

"While the body $\mathcal{B}$ is not to be confused with any of its spatial placements, nevertheless it is available to us only in those placements. For many purposes it is convenient to reflect this fact by using positions in a certain fixed placement as a means of specifying the material points of a body. This *reference placement* may be, but need not be, one actually occupied by the body in the course of its motion." (I have changed the terms *configuration* and *particle* to *placement* and *material point*, respectively.)

In Chapter II of [NLFT] the kinematics of body systems is presented not intrinsically, as in Sects. 3 and 4 above, but only relative to a given placement κ whose frame space coincides with the frame space $\mathcal{F}$ for the motions under consideration. Thus, instead of a motion μ as described in Sects. 3 and 4 above, [NLFT] uses the corresponding *transplacement process* given by

$$\chi(\mathbf{X}, t) = \mu(\kappa^{\leftarrow}(\mathbf{X}), t) \qquad \text{for all} \;\; t \in I, \;\; \mathbf{X} \in \mathrm{Rng}\,(\kappa). \qquad (5.1)$$

(Instead of "transplacement process", the term "family of deformations" is used in [NLFT], but the trouble with this terminology is already alluded to in footnote 4 on p.48 of [NLFT].) The **transplacement gradient** (called "deformation gradient" in [NLFT]) is given by

$$\mathbf{F}(\mathbf{X},t) := \nabla_{\mathbf{X}}\chi(\cdot\,,t) = \mathbf{M}(t)\mathbf{K}^{-1} \qquad \text{for all} \;\; t \in I, \tag{5.2}$$

where $\mathbf{K}$ and $\mathbf{M}(t)$ are given by (4.1) and (4.3) with $X := \kappa^{\leftarrow}(\mathbf{X})$.

If one uses such a reference placement in the formulation of a constitutive law, one must insure that the substance of this law is invariant under changes of reference placement. The best way to insure this is to formulate constitutive laws completely intrinsically, and not use a reference placement at all.

6 Frame-free constitutive laws.

Consider now a continuous body system $\mathcal{B}$. As was discussed in [1], The **principle of material frame-indifference**, as applied to this system, can then be formulated as follows:

The constitutive laws governing the internal interactions between the parts of the system should not depend on whatever external frame of reference is used to describe them.

It is important to note that the principle applies only to *internal* interactions, not to actions of the environment on the system and its parts, because usually the frame of reference employed is actively connected with the environment. For example, if one considers the motion of a fluid in a container, one usually uses the frame of reference determined by the container, which certainly affects the fluid. Inertia should always be considered as an action of the environment on the given system and its parts, and hence its description *does* depend on the frame of reference used. It is also important to note that the principle applies only to *external* frames of refence, not to frames that are constructed from the system itself.

In the past, insuring that this principle is satisfied was not a trivial matter and often required a fair amount of mathematical manipulation. (See, for example, Sect.26 of [NLFT].)

The best way to insure that this principle is satisfied is to formulate constitutive laws completely intrinsically, and not use an external frame of reference at all. For simple materials, one can do this by concentrating on body elements. Such a body element corresponds mathematically to a three-dimensional linear space $\mathcal{T}$ as described in Sect.4. (Since we keep the material point X fixed, we omit the subscript X from now on.) We emphasize again that $\mathcal{T}$ is just a three-dimensional linear space and not an inner product space. A **configuration** of $\mathcal{T}$ is a member of the linear cone $\mathrm{Pos}^{+}(\mathcal{T},\mathcal{T}^{*})$. An inner product is just a configuration that is singled out and made part of the structure if $\mathcal{T}$ were an inner-product space. Here one can still use the concepts and notations usually used in the theory of inner-product spaces, but they must be used only relative to

a given configuration. For example, if $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$ is such a configuration, then a given basis $\mathbf{b} := (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ of $\mathcal{T}$ is $\mathbf{G}$-*orthonormal* if

$$(\mathbf{G}\mathbf{b}_i)\mathbf{b}j = \mathbf{G}(\mathbf{b}_i, \mathbf{b}j) = \delta_{i,j} := \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \qquad \text{for all} \quad i, j \in \{1, 2, 3\} \ . \tag{6.1}$$

A basis that is $\mathbf{G}$-orthonormal necessarily fails to be $\mathbf{G}'$- orthonormal when $\mathbf{G} \neq \mathbf{G}'$. Given *any* basis of $\mathcal{T}$, there is exactly one $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$ that makes it $\mathbf{G}$- orthonormal.

Given any configuration $\mathbf{G} \in \mathcal{G}$, the **orthogonal group** of $\mathbf{G}$ is defined by

$$\mathrm{Orth}\,(\mathbf{G}) := \{\mathbf{A} \in \mathrm{Lis}\,\mathcal{T} \mid \mathbf{A}^\top \mathbf{G}\mathbf{A} = \mathbf{G}\}. \tag{6.2}$$

$\mathrm{Orth}\,(\mathbf{G})$ is not only a subgroup of the linear group $\mathrm{Lis}\,\mathcal{T}$, but even of the **unimodular group**

$$\mathrm{Unim}\,\mathcal{T} := \{\mathbf{A} \in \mathrm{Lis}\,\mathcal{T} \mid \ |\det \mathbf{A}| = 1\}. \tag{6.3}$$

$\mathrm{Unim}\,\mathcal{T}$ includes infinitely many orthogonal groups as subgroups, namely one for each configuration $\mathbf{G}$.

The internal force-interactions in a body system $\mathcal{B}$ can often be described by a *stress* field. Given a motion μ of $\mathcal{B}$, as desribed in Sect.4 above, and a material point $X \in \mathcal{B}$, the Cauchy-stress $\mathbf{T}(t)$ at X and $t \in I$ is a lineon that belongs to the space $\mathrm{Sym}\,\mathcal{V}$, where $\mathcal{V}$ is the translation space of $\mathcal{F}$, which is an inner-product space. Using the motion $\mathbf{M} : I \longrightarrow \mathrm{Lis}\,(\mathcal{T}, \mathcal{V})$ of the given body-element, we associate with the Cauchy-stress $\mathbf{T}(t)$ the **intrinsic stress** defined by

$$\mathbf{S}(t) := \mathbf{M}(t)^{-1}\mathbf{T}(t)\mathbf{M}(t)^{\top -1} \in \ \mathrm{Sym}(\mathcal{T}^*, \mathcal{T}) \quad \text{for all} \quad t \in I. \tag{6.4}$$

A frame-free constitutive law should involve only such intrinsic stresses.

The internal forces in a body system $\mathcal{B}$ can be described in a frame-free manner as follows: Every configuration $d : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{P}$ makes $\mathcal{B}$ a metric set in the sense of Sect.6 of [2]. Using the construction desribed there, one can imbed $\mathcal{B}$ in a Euclidean frame-space $\mathcal{E}_d$ with translation space $\mathcal{V}_d$. For every $X \in \mathcal{B}$ there is a natural isomorphism from $\mathcal{V}_d$ to the tangent space $\mathcal{T}_X$. This isomorphism transports the inner product of $\mathcal{V}_d$ to the configuration $\mathbf{G}$ of the body element $\mathcal{T}_X$ induced by the configuration d of the whole body system $\mathcal{B}$.

If d is the configuration at some time in a deformation process for the whole body system $\mathcal{B}$, then $\mathcal{E}_d$ and $\mathcal{V}_d$ are instantaneous spaces because they also depend on time. A frame-free description of force interactions would use only vectors in the instantaneous space $\mathcal{V}_d$. Stresses would then be members of $\mathrm{Sym}\mathcal{V}_d$. The intrinsic stress can be obtained by transporting the elements of $\mathrm{Sym}\mathcal{V}_\mathrm{d}$ to $\mathrm{Sym}({\mathcal{T}_X}^*, \mathcal{T}_X)$ by using the natural isomorphism from $\mathcal{V}_d$ to $\mathcal{T}_X$ as described above. This means that (6.4) remains valid if $\mathbf{T}(t)$ is interpreted to

be the frame-free stress in $\mathrm{Sym}\mathcal{V}_d$ just described and if $\mathbf{M}(t) \in \mathrm{Lis}\,(\mathcal{T}_X, \mathcal{V}_d)$ is interpreted to be the natural isomorphism described in the previous paragraph.

7. State-space systems.

The Introduction [0] contains the following quote from the preface to the second edition of [NLFT]:

"The Principle of Determinism for the Stress stated on p. 56 of [NLFT] has only limited scope. It should be replaced by a more inclusive principle, using the concept of state rather than a history of infinite duration, as a basic ingredient."

A **state-space system** is a mathematical structure with the following seven ingredients:

(S1) A set $\mathcal{C}$ whose members are called **conditions**.

(S2) A set Π of **processes**, which are one-parameter families of conditions. More precisely, they are mappings of the form

$$\mathbf{P} : I_{\mathbf{P}} \longrightarrow \mathcal{C} \ , \tag{7.1}$$

where $I_{\mathbf{P}}$ is a closed interval whose left endpoint is 0. Thus, $I_{\mathbf{P}}$ is either of the form

$$I_{\mathbf{P}} = [0, d_{\mathbf{P}}] \ \text{ with } \ d_{\mathbf{P}} \in \mathbb{P}\,, \tag{7.2}$$

in which case $d_{\mathbf{P}}$ is called the **duration** of $\mathbf{P}$, or we have $I_{\mathbf{P}} = \mathbb{P}$, in which case we put $d_{\mathbf{P}} := \infty$ and say that $I_{\mathbf{P}}$ has *infinite duration.* One should interpret the value $\mathbf{P}(t)$ to be the *condition at time* t of the system during the given process. We use the notation $\mathbf{P}^i := \mathbf{P}(0)$ for the *initial condition* of $\mathbf{P}$, and, if $\mathbf{P}$ is of finite duration, we use the notation $\mathbf{P}^f := \mathbf{P}(d_{\mathbf{P}})$ for the *final condition* of $\mathbf{P}$.

(S3) A set Σ whose members are called **states**. The set itself is called the **state-space**.

(S4) A mapping

$$\hat{\mathbf{C}} :\longrightarrow \mathcal{C}, \tag{7.3}$$

whose value $\hat{\mathbf{C}}(\sigma)$ is called the *condition of the system when it is in the state* σ.

(S5) An **evolution functional**

$$\hat{\rho} : \{(\mathbf{P}, \sigma) \in \Pi \times \Sigma \mid \hat{\mathbf{C}}(\sigma) = \mathbf{P}^i\} \longrightarrow \Sigma \ . \tag{7.4}$$

Starting with the initial state σ, the final state after applying the process $\mathbf{P}$ to the system is $\hat{\rho}(\mathbf{P}, \sigma)$.

(S6) A set $\mathcal{R}$ whose members are called **responses**.

(S7) a mapping

$$\hat{\mathbf{R}} : \Sigma \longrightarrow \mathcal{R}, \tag{7.5}$$

whose value $\hat{\mathbf{R}}(\sigma)$ represents the response that the system gives when in the state σ.

The set of processes is assumed to satisfy the following two natural axioms:

(P1) Let a process $\mathbf{P}$ and $t_1,\ t_2 \in I_{\mathbf{P}}$ with $t_1 \leq t_2$ be given. We then define the process $\mathbf{P}_{[t_1,t_2]}$, of duration $t_2 - t_1$, by

$$\mathbf{P}_{[t_1,t_2]}(s) := \mathbf{P}(t_1 + s) \qquad \text{for all} \quad s \in [0, t_2 - t_1] \ . \tag{7.6}$$

We say that this process is a **segment** of $\mathbf{P}$. If $\mathbf{P}$ belongs to Π, it is assumed that all of its segments also belong to Π.

(P2) Let two processes $\mathbf{P}_1$ and $\mathbf{P}_2$ be given such that $\mathbf{P}_1$ has finite duration and ${\mathbf{P}_1}^f = {\mathbf{P}_2}^i$. The process $\mathbf{P}_1 * \mathbf{P}_2$, defined by

$$(\mathbf{P}_1 * \mathbf{P}_2)(t) := \begin{cases} \mathbf{P}_1(t) & \text{if } t \in I_{\mathbf{P}_1} \\ \mathbf{P}_2(t - d_{\mathbf{P}_1}) & \text{if } t \in d_{\mathbf{P}_1} + I_{\mathbf{P}_2} \end{cases} \tag{7.7}$$

is then called the **continuation** of $\mathbf{P}_1$ with $\mathbf{P}_2$. Its duration is $d_{\mathbf{P}_1} + d_{\mathbf{P}_2}$. If $\mathbf{P}_1$ and $\mathbf{P}_2$ belong to Π, we assume that $\mathbf{P}_1 * \mathbf{P}_2$ does, too.

The evolution functional $\hat{\rho}$ is assumed to satisfy at least the following three natural axioms.

(E1): Let a state $\sigma \in \Sigma$ and a process $\mathbf{P} \in \Pi$ be given such that $\hat{\mathbf{C}}(\sigma) = \mathbf{P}^i$, We require that

$$\hat{\mathbf{C}}(\hat{\rho}(\sigma, \mathbf{P})) = \mathbf{P}^f \ , \tag{7.8}$$

i.e. that the final state $\hat{\rho}(\sigma, \mathbf{P})$ resulting from the process $\mathbf{P}$ be compatible with the final condition of $\mathbf{P}$.

(E2): Let a state $\sigma \in \Sigma$ and two processes $\mathbf{P_1}$ and $\mathbf{P_2}$ in Π be given such that $\hat{\mathbf{C}}(\sigma) = {\mathbf{P}_1}^i$ and ${\mathbf{P}_2}^i = {\mathbf{P}_1}^f$, as defined in (S2), so that the continuation $(\mathbf{P}_1 * \mathbf{P}_2)$, defined by (7.7), belongs to Π. We require that

$$\hat{\rho}(\sigma, \mathbf{P}_1 * \mathbf{P}_2) = \hat{\rho}(\hat{\rho}(\sigma, \mathbf{P}_1), \mathbf{P}_2) \ . \tag{7.9}$$

(E3): if $\sigma_1, \sigma_2 \in \Sigma$ satisfy $\mathbf{C} := \hat{\mathbf{C}}(\sigma_1) = \hat{\mathbf{C}}(\sigma_2)$ and

$$\hat{\mathbf{R}}(\hat{\rho}(\sigma_1, \mathbf{P})) = \hat{\mathbf{R}}(\hat{\rho}(\sigma_2, \mathbf{P})) \tag{7.10}$$

for all $\mathbf{P} \in \Pi$ with $\mathbf{P}^i = \mathbf{C}$, then we require that $\sigma_1 = \sigma_2$.

The axiom (E3) expresses the assumption that there must be an operational way to distinguish between states. Specifically, if two states are different but produce the same condition, there must be some process which produces a state with two different responses.

Remark 2. I came up with the idea of a state-space system in 1972 in the context of simple material elements as introduced in [N7] and summarized in the next section. State-space systems, although not with the axioms I introduced, have been used before without my being aware of it. (See [W] and the literature cited there.) My paper [N7] had some influence on the work by Coleman, Owen, Del Piero, Silhavy, and others in [CO1], [CO2], [CO3], [D1], [D2], [D3], [DD], [O], [S] [SK], [KS]. See also the references cited in Sect. 4.1 of [S]. The abstract concept of a state-space system presented here seems to be new. ∎

8. Simple material elements.

Here is a sketch of a theory that was developed, in detail, in my paper [N7] of 1972. The structure of a **simple material element** is put on top of a body element as described in Sect.4, i.e. on a given three-dimensional linear space $\mathcal{T}$. This structure will be the structure of a state-space system with the following specifications

(a) The set $\mathcal{C}$ of conditions in (S1) is taken to be a suitable subset $\mathcal{G}$ of the linear cone $\mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$. The members of $\mathcal{G}$ will be called **configurations**.
(b) The set $\mathcal{R}$ of responses in (S5) is taken to be the set $\mathcal{S} := \mathrm{Sym}(\mathcal{T}^*, \mathcal{T})$; its members will be called **intrinsic stresses**.

Everywhere in Sect.6, the symbols $\mathcal{C}$ and $\mathcal{R}$ should now be replaced by $\mathcal{G}$ and $\mathcal{S}$, respectively. Accordingly, $\hat{\mathbf{C}}$ and $\hat{\mathbf{R}}$ should be replaced by $\hat{\mathbf{G}}$ and $\hat{\mathbf{S}}$, respectively. It is assumed, of course, that the axioms (P1),(P2),(E1),(E2), and (E3), are satisfied.

In Sect.11 of [N7] it is explained how one can define, in a natural manner, a topology in the state space Σ and hence consider limits. Let $\mathbf{G} \in \mathcal{G}$ be given. The $\mathbf{G}$-*section*

$$\Sigma_{\mathbf{G}} := \{\sigma \in \Sigma \mid \hat{\mathbf{G}}(\sigma) = \mathbf{G}\} \tag{8.1}$$

of the state space Σ, i.e. the set of all states whose configuration is $\mathbf{G}$, is endowed with a natural Hausdorff topology. The entire state space $\Sigma = \bigcup\{\Sigma_{\mathbf{G}} \mid \mathbf{G} \in \mathcal{G}\}$ is the sum of the topologies of the sections. The mappings $\hat{\mathbf{S}}$ and $\hat{\rho}(\cdot, \mathbf{P})$, for any given process $\mathbf{P}$, are continuous.

Given $\mathbf{G} \in \mathcal{G}$ and $t \in \mathbb{P}$, we call the process $\mathbf{G}_{(t)} : [0, t] \longrightarrow \mathcal{G}$, defined by

$$\mathbf{G}_{(t)}(s) := \mathbf{G} \qquad \text{for all} \quad s \in [0, t] \ , \tag{8.2}$$

the **freeze** at $\mathbf{G}$ of duration t. We now add an additonal axiom:

(F) For every $\mathbf{G} \in \mathcal{G}$ and every $t \in \mathbb{P}$, the freeze $\mathbf{G}_{(t)}$ belongs to Π, and for every state $\sigma \in \Sigma$ the limit

$$\hat{\lambda}(\sigma) := \lim_{t \to \infty} \hat{\rho}(\sigma, \hat{\mathbf{G}}(\sigma)_{(t)}) \tag{8.3}$$

exists.

The members of the range of the mapping $\hat{\lambda} : \Sigma \longrightarrow \Sigma$ defined by (8.3) are called **relaxed states**. We denote this range by $\Sigma_{\rm rel}$.

We say that the given material element is **semi-elastic** if two different relaxed states cannot produce the same configuration, i.e. if $\hat{\mathbf{G}}(\sigma_1) = \hat{\mathbf{G}}(\sigma_2)$ with $\sigma_1, \sigma_2 \in \Sigma_{\rm rel}$ can happen only when $\sigma_1 = \sigma_2$. Theorem 16.1 in [N7] shows that the theory of semi-elastic materials is but a reformulation of the older theory of simple materials with memory first formulated by me in my paper [N12] of 1958 and described in [NLFT].

We say that the given material element is **elastic** if it is semi-elastic and if all of its states are relaxed. If this is the case, the state space can be identified with $\mathcal{G}$ and the evolution functional becomes trivial.

Let two simple material elements with ingredients $(\mathcal{T}_1, \mathcal{G}_1, \Pi_1, \Sigma_1, \hat{\mathbf{G}}_1, \hat{\rho}_1, \hat{\mathbf{S}}_1)$ and $(\mathcal{T}_2, \mathcal{G}_2, \Pi_2, \Sigma_2, \hat{\mathbf{G}}_2, \hat{\rho}_2, \hat{\mathbf{S}}_2)$ be given. A **material isomorphism** between these elements is then induced by a linear isomorphism $\mathbf{A} \in \mathrm{Lis}\,(\mathcal{T}_1, \mathcal{T}_2)$ with the following properties:

(i) $\mathbf{A}^\top \mathcal{G}_2 \mathbf{A} = \mathcal{G}_1$,
(ii) $\mathbf{A}^\top \Pi_2 \mathbf{A} = \Pi_1$, and
(iii) there is an invertible mapping

$$\iota_{\mathbf{A}} : \Sigma_1 \longrightarrow \Sigma_2 \tag{8.4}$$

such that, for every $\sigma \in \Sigma_1$, the equations

$$\hat{\mathbf{G}}_1(\sigma) = \mathbf{A}^\top \hat{\mathbf{G}}_2(\iota_{\mathbf{A}}(\sigma))\mathbf{A} \ , \tag{8.5}$$

$$\hat{\mathbf{S}}_2(\iota_{\mathbf{A}}(\sigma)) = \mathbf{A}\hat{\mathbf{S}}_1(\sigma)\mathbf{A}^\top \ , \tag{8.6}$$

and

$$\iota_{\mathbf{A}}(\hat{\rho}_1(\sigma, \mathbf{A}^\top \mathbf{P}\mathbf{A})) = \rho_1(\iota_{\mathbf{A}}(\sigma), \mathbf{P}) \tag{8.7}$$

hold for all $\mathbf{P} \in \Pi_2$ such that $\hat{\mathbf{G}}_1(\sigma) = \mathbf{A}^\top \mathbf{P}^i \mathbf{A}$.

In Sect.9 of [N7] it was shown that there can be at most one $\iota_{\mathbf{A}}$ with the properties just described. It is a homeomorphism between the topological spaces Σ_1 and Σ_2.

We now deal again with a single material element. Then $\mathbf{A} \in \mathrm{Lis}\,\mathcal{T}$ is called a **symmetry** of the element if it induces a material isomorphism from the element to itself. These symmetries form a subgroup g of $\mathrm{Lis}\,\mathcal{T}$, called the **symmetry group** of the element. It acts on the state space Σ by an action $\iota : g \longrightarrow \mathrm{Perm}\,\Sigma$ that has the following properties:

$$\hat{\mathbf{G}}(\sigma) = \mathbf{A}^\top \hat{\mathbf{G}}(\iota_{\mathbf{A}}(\sigma))\mathbf{A} \tag{8.8}$$

and

$$\hat{\mathbf{S}}(\iota_{\mathbf{A}}(\sigma)) = \mathbf{A}\hat{\mathbf{S}}(\sigma)\mathbf{A}^\top \tag{8.9}$$

hold for all $\sigma \in \Sigma$ and all $\mathbf{A} \in g$, and

$$\iota_{\mathbf{A}}(\hat{\rho}(\sigma, \mathbf{P})) = \hat{\rho}(\iota_{\mathbf{A}}(\sigma), (\mathbf{A}^{\top})^{-1}\mathbf{P}\mathbf{A}^{-1}) \tag{8.10}$$

holds for all $\sigma \in \Sigma$, all $\mathbf{A} \in g$, and all deformation processes $\mathbf{P}$ such that $\hat{\mathbf{G}}(\sigma) = \mathbf{P}^i$. (For an explanation of the concept of an action of a group on a set, see Sect.31 [FDS].)

Again, let a state $\sigma \in \Sigma$ be given. The set

$$g_\sigma := \{\mathbf{A} \in g \mid \ \iota_{\mathbf{A}}(\sigma) = \sigma\} \tag{8.11}$$

of all symmetries that leave σ invariant is a subgroup of g; it is called the **symmetry group of the state** σ. g_σ is also a subgroup of the orthogonal group of the configuration $\hat{\mathbf{G}}(\sigma)$, so that

$$g_\sigma \ \subset \ g \ \cap \ \mathrm{Orth}\,(\hat{\mathbf{G}}(\sigma)) \ . \tag{8.12}$$

We say that the state σ is **isotropic** if $g_\sigma = \mathrm{Orth}\,(\hat{\mathbf{G}}(\sigma))$.

Given any symmetry $\mathbf{A} \in g$, the symmetry groups of the states σ and $\iota_{\mathbf{A}}(\sigma)$ are conjugate. In fact, we have

$$g_{\iota_{\mathbf{A}}(\sigma)} = \mathbf{A} g_\sigma \mathbf{A}^{-1} \ . \tag{8.13}$$

Definition 1. *We say that the element is* **semi-fluid** *if its symmetry group is the full unimodular group, i.e. if* $g = \mathrm{Unim}\,\mathcal{T}$ *(see (6.3)). We say that the element is* **fluid** *if it is both semi-fluid and semi-elastic in the sense defined above. If the element is semi-fluid [fluid] and incompressible in the sense of Sect.8 of [N7] we say that it is* **semi-liquid** *[***liquid***].* *)

Remark 3. The theory summarized above has seen only few direct applications in the past 30 years. Notable among them are in the *Theory of Inelastic Behavior of Materials* presented by Šilhavý and Kratochvíl in [SK] and [KS] and in the paper *On the Elastic-Plastic Material Element* by Del Piero in [D1] and [D2]. This work corroborates the following claim in the Preface to the Second Edition in [NLFT]: "The new concept of material makes it possible also to include theories of 'plasticity' in the general framework, and one can now do much more than 'refer the reader to the standard treatises' as we suggested on p.11 of the Introduction." ∎

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*) The term "fluid" was already introduced in [N7]. Using "liquid" for "incompressible fluid" is compatible with the dictionary definition: "liquid refers to a substance that flows readily ... but retains its independent volume ; fluid applies to any substance that flows" (Webster's New World Dictionary, Second College Edition). The prefix *semi* is used to indicate that flow may occur only after sufficient force is applied, as is illustrated in Sect.6 of the sequel [4] to this paper.

A theory of simple materials that includes not only mechanics but also thermodynamics has not yet been fully developed, although the work of Coleman and Owen in Part II of [CO1] may be a good beginning. Unfortunetely, their description is not frame-free.

A thermomechanical process of a body system $\mathcal{B}$, in a frame-free description, should involve, in addition to a deformation process, a *temperature process*

$$\hat{\theta} : \mathcal{B} \times I \longrightarrow \mathbb{P}^{\times}, \tag{8.14}$$

whose value $\hat{\theta}(X,t)$ is the absolute temperature at the material point $X \in \mathcal{B}$ at time $t \in I$. Assuming that $\hat{\theta}$ is differentiable, we can consider the *temperature gradient*

$$\mathbf{g} := \nabla_X \hat{\theta}(\cdot, t) \in \mathrm{Lin}\,(\mathbb{R}, \mathcal{T}_X) = \mathcal{T}_X{}^* \ . \tag{8.15}$$

As indicated in Sect.21 (b) of [N7], a thermomechanical structure of a simple material element should be a state-space structure with the following specifications:

(A) The set $\mathcal{C}$ of conditions in (S1) is taken to be of the form $\mathcal{C} := \mathcal{G} \times \mathbb{P}^{\times} \times \mathcal{T}^*$, where $\mathcal{G}$ is a set of configurations as described in (a) above. The triple $\mathbf{C} := (\mathcal{G}, \theta, \mathbf{g}) \in \mathcal{C}$ gives the configuration, the temperature, and the temperature gradient of the condition $\mathbf{C}$.

(B) The set $\mathcal{R}$ of responses in (S5) is taken to be of the form, $\mathcal{R} := \mathcal{S} \times \mathbb{P} \times \mathbb{R} \times \mathcal{T}$, where $\mathcal{S} := \mathrm{Sym}(\mathcal{T}^*, \mathcal{T})$ as in (b) above. The quadruple $\mathbf{R} := (\mathbf{S}, \epsilon, \eta, \mathbf{q})$ gives the intrinsic stress, the internal energy per unit volume, the entropy per unit volume, and the intrinsic heat flux of the response $\mathbf{R}$.

A formula for the rate of entropy production, like (21.4) in [N7], should play a fundamental role in developing the theory.

9 Material elements of grade two.

Not all situations in the real world can be modelled adequately by a theory of *simple* material elements. For example, models intended to describe the behavior of liquid crystals often require a theory based on body elements given not just by the tangent space $\mathcal{T}$ but a second order tangent structure to a point of the C^2-manifold describing the continuous body system, as explained in Sect.3 above, if such a theory is formulated in a frame-free way. Such second order tangent structures are quite complicated. Following [N13], they involve the following ingredients and axioms: Here are the ingredients:

(1) The tangent space $\mathcal{T}$,
(2) A linear space $\mathcal{S}$, called the *shift space*,
(3) A linear mapping $\mathbf{I} : \mathrm{Lin}\,\mathcal{T} \longrightarrow \mathcal{S}$,
(4) A linear mapping $\mathbf{P} : \mathcal{S} \longrightarrow \mathcal{T}$,
(5) A bilinear mapping $\mathbf{B} \in \mathrm{Skew}_2(\mathcal{S} \times \mathcal{S}, \mathcal{T})$, called the *shift bracket*.

Here are the axioms:

(6) $\mathbf{I}$ is injective, $\mathbf{P}$ is surjective, and $\operatorname{Null}\mathbf{P} = \operatorname{Rng}\mathbf{I}$,
(7) The linear mapping

$$\mathbf{s} \longmapsto \mathbf{B}(\mathbf{s}, \cdot) \quad \in \ \operatorname{Lin}(\mathcal{S}, \operatorname{Lin}(\mathcal{S}, \mathcal{T})) \tag{9.1}$$

is injective.

The content of (1)-(4) with the axiom (6) is often expressed by saying that

$$\{\mathbf{0}\} \longrightarrow \operatorname{Lin}\mathcal{T} \xrightarrow{\mathbf{I}} \mathcal{S} \xrightarrow{\mathbf{P}} \mathcal{T} \longrightarrow \{\mathbf{0}\} \tag{9.2}$$

is a *short exact sequence*

If $\mathcal{T}$ has dimension 3, then $\operatorname{Lin}\mathcal{T}$ has dimension $3^2 = 9$ and $\mathcal{S}$ has dimension $3^2 + 3 = 12$.

We call the structure thus described a **second grade body element**.

A linear mapping $\mathbf{K} \in \operatorname{Lin}(\mathcal{T}, \mathcal{S})$ is called a **connector** if it is a right inverse of $\mathbf{P}$, i.e. if

$$\mathbf{PK} = \underline{1}_{\mathcal{T}} \ . \tag{9.3}$$

The set of all connectors, which we denote by $\operatorname{Conn}(\mathcal{T}, \mathcal{S})$, is a flat in the linear space $\operatorname{Lin}(\mathcal{T}, \mathcal{S})$. The *torsion* of a connector $\mathbf{K}$ is defined by

$$\operatorname{Tor}(\mathbf{K}) := -\mathbf{B} \circ (\mathbf{K} \times \mathbf{K}) \in \operatorname{Skew}_2(\mathcal{T} \times \mathcal{T}, \mathcal{T}) \ . \tag{9.4}$$

We say that a given connector $\mathbf{K}$ is *symmetric* if $\operatorname{Tor}(\mathbf{K}) = \mathbf{0}$. The set of all symmetric connectors will be denoted by $\operatorname{Sconn}(\mathcal{T}, \mathcal{S})$. It is also a flat in $\operatorname{Lin}(\mathcal{T}, \mathcal{S})$.

A configuration of a continuous body-system as described in Sect.3 above is a metric that induces a Riemannian structure on the body-manifold. This structure not only assigns a "metric tensor" to each point X, here called the *configuration* of the body element $\mathcal{T}$, but also what is often called a "symmetric affine connection". Such an affine connection can be viewed as a field of symmetric connectors as defined above, one for each point in the body system. Thus, a condition for this system is reasonably defined to be a pair $(\mathbf{G}, \mathbf{K})$, where $\mathbf{G} \in \operatorname{Pos}^+(\mathcal{T}, \mathcal{T}^*)$ as in Sect.8, and where $\mathbf{K} \in \operatorname{Sconn}(\mathcal{T}, \mathcal{S})$.

The structure of a **material element of grade two** is put on top of a second grade body element as described above. This structure will be the structure of a state-space system, as described in Sect.6, with the following specifications:

(a) The set $\mathcal{C}$ of conditions in (S1) is taken to be a suitable subset of the set $\operatorname{Pos}^+(\mathcal{T}, \mathcal{T}^*) \times \operatorname{Sconn}(\mathcal{T}, \mathcal{S})$. Each member of $\mathcal{C}$ is a pair $(\mathbf{G}, \mathbf{K})$ as described above.
(b) The set $\mathcal{R}$ of responses in (S5) is taken to be the set $\mathcal{R} := \operatorname{Sym}(\mathcal{T}^*, \mathcal{T}) \times \operatorname{Skew}_2(\mathcal{T}^* \times \mathcal{T}^*, \mathcal{T})$. Each member of $\mathcal{R}$ is a pair $(\mathbf{S}, \mathbf{C})$,

where $\mathbf{S} \in \mathrm{Sym}(\mathcal{T}^*, \mathcal{T})$ is an **intrinsic stress**, and $\mathbf{C} \in \mathrm{Skew}_2(\mathcal{T}^* \times \mathcal{T}^*, \mathcal{T})$ is an **intrinsic couple stress**.

It is my conjecture that the internal interactions in materials of grade 2 cannot be described by forces alone but must also include couples.

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[4] The Theory of Simple Semi-Liquids, a Conceptual Framework for Rheology

0. Introduction.

In this paper, the framework of the New Theory of Simple Materials introduced in [N7] in 1972 and summarized in [3] is used to give a careful analysis of the concept of a *monotonous state* (called a "state of monotonous flow" in [N7]) and to introduce the narrower concept of a *uni-monotonous state*, of which a *state of simple shearing* is a special case. New is also a detailed analysis of material elements that are *semi-liquid* according to Def.1 of [3].

In the last two sections of this paper, it will be shown how the theory of semi-liquids generalizes the now conventional theory of incompressible simple fluids. In particular, it will be shown how the entire theory of viscometric flows treated in [CMN] can be extended to semi-liquids. In fact, the present paper could be the basis for a radical update of [CMN] as well as of Chapter E of [NLFT]. The examples of Poiseuille flow and Couette flow will be analyzed explicitly in the last section here. It should be a fairly easy exercise to produce similar analyses for helical flow, cone and plate flow, torsional flow, and the other examples of visometric flows treated in Chapter IV of [CMN]. I believe that the theory of semi-liquids is a good mathematical model for many materials considered by rheologists, toothpaste being a familiar example.

The theory of semi-liquids presented here is conceptual, axiomatic, and very general. It can serve as a framework for more specific and more special theories. In the sequel [5] to this paper, I will present such a special theory, which I call the theory of *nematic semi-liquids*. It may furnish a mathematical model for certain aspects of the flow of nematic liquid crystals.

To understand the mathematical background used in this paper, the reader should be thoroughly familiar with the concepts, terminology, and notation used [3] and in [FDS].

For later use, we record here the rule for the gradient of the determinant function $\det : \mathrm{Lin}\,\mathcal{T} \longrightarrow \mathbb{R}$:

$$(\nabla_{\mathbf{L}} \det)\mathbf{M} = (\det \mathbf{L})\mathrm{tr}(\mathbf{M}\mathbf{L}^{-1}) \qquad \text{for all} \;\; \mathbf{L} \in \mathrm{Lis}\,\mathcal{T}\ ,\ \mathbf{M} \in \mathrm{Lin}\,\mathcal{T}\ . \tag{0.1}$$

1. Monotonous processes.

We assume now that body elements $\mathcal{T}_1$ and $\mathcal{T}_2$ and corresponding deformation processes (as defined in Sect.4 of [N3]) $\mathbf{P}_1$ and $\mathbf{P}_2$ are given. We say that $\mathbf{P}_1$ and $\mathbf{P}_2$ are **congruent** if $\mathbf{P}_1$ and $\mathbf{P}_2$ have the same duration $d := d_{\mathbf{P}_1} = d_{\mathbf{P}_2}$ and if there is a linear isomorphism $\mathbf{A} \in \mathrm{Lis}\,(\mathcal{T}_1, \mathcal{T}_2)$ such that

$$\mathbf{P}_1(t) = \mathbf{A}^{\top}\mathbf{P}_2(t)\mathbf{A} \quad \text{for all} \;\; t \in [0, d]. \tag{1.1}$$

Remark 1: This notion of congruence is analogous to congruence of curves in a Euclidean space. Intuitively, if one takes still pictures of two congruent curves,

one cannot tell from the pictures which is which. Analogously, if one makes motion pictures of two congruent deformation processes, one cannot tell which is which by showing the movies. ▮

We assume that now that a body element $\mathcal{T}$ is given.

Definition 1. *We say that a deformation process* $\mathbf{P}$ *of* $\mathcal{T}$ *is* **monotonous** *if any two segments of* $\mathbf{P}$ *of equal duration are congruent.*

Remark 2: An idea that led to the concept of a monotonous process, as defined here, was first introduced by Coleman in his paper [C] in 1962, but he used the term "substantially stagnant motion". In the same paper, he also introduced the concept and the term "viscometric flow". Shortly thereafter, in [N14], I found a description of monotonous flows in terms of lineonic exponentials. (I then used the term "motion with constant stretch history".) In [N15] in 1972, I gave a rigorous proof of the Representation Theorem for Monotonous Processes stated below, and started using the adjective "monotonous", because the earlier terminology was not suitable for the frame-free approach I employed from then on. ▮

A trivial example of a monotonous process is a *freeze*, which is a process whose value-configuration is constant (see Sect.8 of [3]). The following basic theorem characterizes all monotonous processes in terms of the lineonic exponential function $\exp : \mathrm{Lin}\,\mathcal{T} \longrightarrow \mathrm{Lin}\,\mathcal{T}$, which is defined and discussed in Sect.612 of [FDS]. It is characterized by the following property: Given $\mathbf{E} \in \mathrm{Lin}\,\mathcal{T}$ and defining $\mathbf{F} : \mathbb{R} \longrightarrow \mathrm{Lin}\,\mathcal{T}$ by

$$\mathbf{F}(t) := \exp(t\mathbf{E}) \qquad \text{for all} \qquad t \in \mathbb{P} \ , \tag{1.2}$$

we have

$$\mathbf{F}^{\bullet} = \mathbf{E}\mathbf{F} = \mathbf{F}\mathbf{E} \ \text{ and } \ \mathbf{F}(0) = \mathbf{1}_{\mathcal{T}} \ . \tag{1.3}$$

Also, we have

$$\exp(t\,(\mathbf{A}\mathbf{E}\mathbf{A}^{-1}) = \mathbf{A}\exp(t\mathbf{E})\,\mathbf{A}^{-1} \ \text{ for every } \mathbf{A} \in \mathrm{Lis}\,\mathcal{T} \text{ and for all } t \in \mathbb{R}. \tag{1.4}$$

Representation Theorem for Monotonous Processes: *A given deformation process* $\mathbf{P}$ *is monotonous if and only if there is a lineon* $\mathbf{E} \in \mathrm{Lin}\,\mathcal{T}$ *such that*

$$\mathbf{P}(t) = \exp(t\mathbf{E}^{\top})\mathbf{P}^{i}\exp(t\mathbf{E}) \qquad \text{for all} \qquad t \in I_{\mathbf{P}}. \tag{1.5}$$

If this is the case, we say that $\mathbf{P}$ *is a monotonous process* **of exponent** $\mathbf{E}$.

The proof of this Theorem, which is highly non-trivial, can be found in [N15].

Remark 3: One can define *monotonous curves* in a three-dimensional Euclidean space in a manner that is analogous to the definition above. It can easily be proved that a curve is monotonous in this sense if and only if it has constant curvature and torsion. The only such curves are straight, circular, or helical. ▮

We record the following result for later use.

Proposition 1. *Let a monotonous process* $\mathbf{P}$ *of exponent* $\mathbf{E}$ *and* $\mathbf{A} \in \mathrm{Lis}\,\mathcal{T}$ *be given. Then* $(\mathbf{A}^{-1})^\top \mathbf{P}\mathbf{A}^{-1}$ *is a monotonous process of exponent* $\mathbf{A}\mathbf{E}\mathbf{A}^{-1}$. *Its initial configuration is* $(\mathbf{A}^{-1})^\top \mathbf{P}^i \mathbf{A}^{-1}$.

Proof: Let $t \in I_{\mathbf{P}}$ be given. By (1.5) we have

$$(\mathbf{A}^{-1})^\top \mathbf{P}(t)(\mathbf{A}^{-1}) = (\mathbf{A}^{-1})^\top \exp(t\mathbf{E}^\top)\mathbf{P}^i \exp(t\mathbf{E})\mathbf{A}^{-1}.$$

Using (1.4), it follows that

$$(\mathbf{A}^{-1})^\top \mathbf{P}(t)(\mathbf{A}^{-1}) = \exp(t\,(\mathbf{A}\mathbf{E}\mathbf{A}^{-1})^\top)(\mathbf{A}^{-1})^\top \mathbf{P}^i(\mathbf{A}^{-1})\exp(t\,(\mathbf{A}\mathbf{E}\mathbf{A}^{-1})). \quad (1.6)$$

Since $t \in I_{\mathbf{P}}$ was arbitrary and since $(\mathbf{A}^{-1})^\top \mathbf{P}^i \mathbf{A}^{-1} = ((\mathbf{A}^{-1})^\top \mathbf{P}\mathbf{A}^{-1})^i$, the assertion follows from (1.6) and the Representation Theorem. ∎

Let $\mathbf{E} \in \mathrm{Lin}\,\mathcal{T}$ be given. We denote the subgroup of $\mathrm{Lis}\,\mathcal{T}$ that consists of all $\mathbf{A} \in \mathrm{Lis}\,\mathcal{T}$ that commute with $\mathbf{E}$ by

$$\mathrm{Com}\,\mathbf{E} := \{\mathbf{A} \in \mathrm{Lis}\,\mathcal{T} \mid \mathbf{A}\mathbf{E}\mathbf{A}^{-1} = \mathbf{E}\}. \quad (1.7)$$

2. Monotonous states.

We now assume that a simple material element, in the sense of Sect.8 of [3], is given, and consider its symmetry group $\mathcal{g}$

Definition 2. *Let a state* $\sigma \in \Sigma$ *and a lineon* $\mathbf{E} \in \mathrm{Lin}\,\mathcal{T}$ *be given. We say that* σ *is a* **monotonous state** *and that the* **exponent** *of* σ *is* $\mathbf{E}$ *if*

$$\exp(s\mathbf{E}) \in \mathcal{g} \qquad \text{for all} \qquad s \in \mathbb{R} \quad (2.1)$$

and if

$$\iota_{\exp(t\mathbf{E})}(\hat{\rho}(\sigma, \mathbf{P}_{[0,t]})) = \sigma \qquad \text{for all} \qquad t \in \mathbb{P}\ , \quad (2.2)$$

where $\mathbf{P}$ *is the monotonous process of exponent* $\mathbf{E}$ *and of infinite duration which satisfies* $\mathbf{P}^i = \hat{\mathbf{G}}(\sigma)$, *i.e. the process defined by (1.5) with* $\mathbf{P}^i := \hat{\mathbf{G}}(\sigma)$ and $I_{\mathbf{P}} := \mathbb{P}$.

Remark 4: If $\mathcal{g}$ is a Lie-group, then (2.1) states that $\mathbf{E}$ belongs to the Lie-algebra of $\mathcal{g}$. ∎

Consider the case when $\mathbf{E} := \mathbf{0}$. Then the process $\mathbf{P}$ of Def.2 becomes the freeze of infinite duration at $\mathbf{G} := \hat{\mathbf{G}}(\sigma)$ and (2.2) reduces to

$$\hat{\rho}(\sigma, \mathbf{G}_{(t)}) = \sigma \qquad \text{for all} \qquad t \in \mathbb{P}\ , \quad (2.3)$$

where $\mathbf{G}_{(t)}$ is the freeze of duration t at $\mathbf{G}$.

Now, by Prop.12.1 of [N7], a state $\sigma \in \Sigma_{\mathbf{G}}$ is a *relaxed state*, as defined in Sect.8 of [3], if and only if (2.3) holds. Hence the monotonous states whose exponent is $\mathbf{0}$ are just the relaxed states.

We now assume that $\mathbf{G} \in \mathcal{G}$ and $\mathbf{E} \in \mathrm{Lin}\,\mathcal{T}$ are given.

Proposition 2. *Let* $\mathbf{P}$ *be the monotonous process of exponent* $\mathbf{E}$*, of infinite duration, and of initial configuration* $\mathbf{P}^i = \mathbf{G}$*. Assume that (2.1) holds. Let* $\tau_0 \in \Sigma_{\mathbf{G}}$ *be given and put*

$$\tau_t := \iota_{\exp(t\mathbf{E})}(\hat{\rho}(\tau_0, \mathbf{P}_{[0,t]})) \qquad \text{for all} \quad t \in I\!P \ . \tag{2.4}$$

Then $\tau_t \in \Sigma_{\mathbf{G}}$ *for all* $t \in I\!P$*. If the family* $(\tau_t \mid t \in I\!P)$ *converges, then its limit not only belongs to* $\Sigma_{\mathbf{G}}$ *but is a monotonous state of exponent* $\mathbf{E}$*.*

Proof: Let $t \in \mathbb{P}$ be given. We apply $\hat{\mathbf{G}}$ to both sides of (2.4). Using (8.10) of [3] with $\mathbf{A}$ replaced by $\exp(t\mathbf{E})$, then using axiom [E1] of Sect.7 of [3], and finally (1.5), we obtain

$$\hat{\mathbf{G}}(\tau_t) = \hat{\mathbf{G}}(\hat{\rho}(\iota_{\exp(t\mathbf{E})}(\tau_0), \exp(-t\mathbf{E}^\top)\mathbf{P}_{[0,t]} \exp(-t\mathbf{E}))) =$$

$$= (\exp(-t\mathbf{E}^\top)\mathbf{P}_{[0,t]} \exp(-t\mathbf{E}))^f = \exp(-t\mathbf{E}^\top)\mathbf{P}(t) \exp(-t\mathbf{E}) = \mathbf{P}(0) = \mathbf{P}^i = \mathbf{G} \ ,$$

which means that $\tau_t \in \Sigma_{\mathbf{G}}$.

Let $t, r \in \mathbb{P}$ be given. It follows from (7.7), (7.9) of [N3], and from (1.5) above that

$$\mathbf{P}_{[0,r+t]} = \mathbf{P}_{[0,r]} * \mathbf{P}_{[r,r+t]} = \mathbf{P}_{[0,r]} * \exp(r\mathbf{E}^\top)\mathbf{P}_{[0,t]} \exp(r\mathbf{E}) \ . \tag{2.5}$$

Using first the axiom [E2] of Sect.7 and then (8.10) of [3], with $\mathbf{A}$ replaced by $\exp(-r\mathbf{E}) = (\exp(r\mathbf{E})^{-1}$, we conclude from (2.5) that

$$\hat{\rho}(\tau_0, \mathbf{P}_{[0,r+t]}) = \hat{\rho}(\hat{\rho}(\tau_0, \mathbf{P}_{[0,r]}), \exp(r\mathbf{E}^\top)\mathbf{P}_{[0,t]} \exp(r\mathbf{E})) =$$
$$= (\iota_{\exp(r\mathbf{E})})^{\leftarrow}(\hat{\rho}(\iota_{\exp(r\mathbf{E})}(\hat{\rho}(\tau_0, \mathbf{P}_{[0,r]})), \mathbf{P}_{[0,t]}) \ . \tag{2.6}$$

Since $\iota_{\exp((r+t)\mathbf{E})} = \iota_{\exp(t\mathbf{E})} \circ \iota_{\exp(r\mathbf{E})}$, it follows from (2.6) that

$$\iota_{\exp((r+t)\mathbf{E})}\hat{\rho}(\tau_0, \mathbf{P}_{[0,r+t]}) = \iota_{\exp(t\mathbf{E})}(\hat{\rho}(\iota_{\exp(r\mathbf{E})}(\hat{\rho}(\tau_0, \mathbf{P}_{[0,r]})), \mathbf{P}_{[0,t]}) \ . \tag{2.7}$$

Assume now that the family $(\tau_t \mid t \in \mathbb{P})$ defined by (2.4) converges and denote its limit by σ. Since we deal with a topology in $\Sigma_{\mathbf{G}}$, the limit σ must belong to $\Sigma_{\mathbf{G}}$. Taking the limit $r \to \infty$ in (2.7) and using the continuity properties of $\hat{\rho}$ and $\iota_{\exp(r\mathbf{E})}$, we obtain

$$\sigma = \iota_{\exp(t\mathbf{E})}(\hat{\rho}(\sigma, \mathbf{P}_{[0,t]})) \ .$$

Since $t \in \mathbb{P}$ was arbitrary, it follows from Def.2 that σ is indeed a monotonous state. ∎

In the case when $\mathbf{E} := \mathbf{0}$, the definition (2.4) reduces to $\tau_t := \hat{\rho}(\tau_0, \mathbf{G}_{(t)})$. By Axiom (F) in Sect.8 of [3], the family $(\tau_t \mid t \in \mathbb{P})$ then converges to the relaxed state $\hat{\lambda}(\tau_0)$ obtained from τ_0 by the relaxation mapping $\hat{\lambda} : \Sigma \longrightarrow \Sigma$ defined by (8.3).

Definition 3. *Let a monotonous state σ be given. Assume that its exponent is $\mathbf{E}$ and that its configuration is $\mathbf{G}$. We say that σ is* **stable** *if there is a neighborhood Υ of σ in $\Sigma_{\mathbf{G}}$ with the following property: For every $\tau_0 \in \Upsilon$ the family $(\tau_t \mid t \in \mathbb{P})$, defined by (2.4) in Prop.2, converges to σ.*

Proposition 3. *Let a monotonous state σ and $\mathbf{A} \in \mathfrak{g}$ be given. Then $\iota_{\mathbf{A}}(\sigma)$ is again a monotonous state, and if $\mathbf{E}$ is the exponent of σ then $\mathbf{AEA}^{-1}$ is the exponent of $\iota_{\mathbf{A}}(\sigma)$. Moreover, if σ is stable, so is $\iota_{\mathbf{A}}(\sigma)$.*

Proof: Put $\mathbf{G} := \hat{\mathbf{G}}(\sigma)$ and consider the monotonous process $\mathbf{P}$ of infinite duration, with exponent $\mathbf{E}$, and with $\mathbf{P}^i = \mathbf{G}$. Put

$$\mathbf{E}' := \mathbf{AEA}^{-1}, \qquad \mathbf{P}' := (\mathbf{A}^{-1})^{\top}\mathbf{PA}^{-1} \ . \tag{2.8}$$

By Prop.1 $\mathbf{P}'$ is the monotonous process of infinite duration, with exponent $\mathbf{E}'$ and initial configuration $\mathbf{P}^i = (\mathbf{A}^{-1})^{\top}\mathbf{GA}^{-1}$.

Let $t \in \mathbb{P}$ be given. It follows from $(2.8)_2$ and from (8.10) in [3], with $\mathbf{P}$ replaced by $\mathbf{P}_{[0,t]}$, that

$$\iota_{\mathbf{A}}(\hat{\rho}(\tau, \mathbf{P}_{[0,t]})) = \hat{\rho}(\iota_{\mathbf{A}}(\tau), \mathbf{P}'_{[0,t]}) \quad \text{for all} \quad \tau \in \Sigma_{\mathbf{G}} \ . \tag{2.9}$$

By $(2.8)_1$ and (1.4) we have $\exp(t\mathbf{E}') = \mathbf{A}\exp(t\mathbf{E})\mathbf{A}^{-1}$ and hence

$$\iota_{\exp(t\mathbf{E}')} = \iota_{\mathbf{A}} \circ \iota_{\exp(t\mathbf{E})} \circ (\iota_{\mathbf{A}})^{\leftarrow} \ . \tag{2.10}$$

Applying the mapping (2.10) to (2.9), we see that

$$\iota_{\mathbf{A}}(\iota_{\exp(t\mathbf{E})}(\hat{\rho}(\tau, \mathbf{P}_{[0,t]}))) = \iota_{\exp(t\mathbf{E}')}(\hat{\rho}(\iota_{\mathbf{A}}(\tau), \mathbf{P}'_{[0,t]})) \quad \text{for all} \quad \tau \in \Sigma_{\mathbf{G}} \ . \tag{2.11}$$

In the case when $\tau := \sigma$, it follows from (2.2) that (2.11) reduces to

$$\iota_{\mathbf{A}}(\sigma) = \iota_{\exp(t\mathbf{E}')}(\hat{\rho}(\iota_{\mathbf{A}}(\sigma), \mathbf{P}'_{[0,t]})) \ .$$

Since $t \in \mathbb{P}$ was arbitrary, this means, by Def.2, that $\iota_{\mathbf{A}}(\sigma)$ is indeed a monotonous state.

Now assume that the monotonous state σ is stable and consider a neighborhood Υ of σ in accord with Def.3. Since $\iota_{\mathbf{A}}$ is a homeomorphism, the image $(\iota_{\mathbf{A}})_{>}(\Upsilon)$ is a neighborhood of $\iota_{\mathbf{A}}(\sigma)$. Thus, a given member of this image-neighborhood must be of the form $\iota_{\mathbf{A}}(\tau_0)$, where $\tau_0 \in \Upsilon$. By Def.3, the family $(\tau_t \mid t \in \mathbb{P})$ defined by (2.4) must converge to σ. Using (2.9) with $\tau := \tau_0$ for every $t \in \mathbb{P}$, it follows from (2.4) that the family $(\tau'_t \mid t \in \mathbb{P})$ defined by

$$\tau'_t := \iota_{\exp(t\mathbf{E}')}(\hat{\rho}(\iota_{\mathbf{A}}(\tau_0), \mathbf{P}'_{[0,t]})) \qquad \text{for all} \quad t \in \mathbb{P}$$

satisfies $\iota_{\mathbf{A}}(\tau_t) = \tau'_t$ for all $t \in \mathbb{P}$. By the continuity of $\iota_{\mathbf{A}}$, we see that $(\tau'_t \mid t \in \mathbb{P})$ must converge to $\iota_{\mathbf{A}}(\sigma)$. Therefore, $\iota_{\mathbf{A}}(\sigma)$ is again stable. ∎

3. Uni-monotonous states, states of simple shearing.

We assume again that a simple material element is given.

Definition 4. *Let a state $\sigma \in \Sigma$ be given and denote its configuration by $\mathbf{G} := \hat{\mathbf{G}}(\sigma)$. We say that σ is a* **uni-monotonous** *state if*
(i) σ is a stable monotonous state, and
(ii) there is no other stable monotonous state whose configuration is $\mathbf{G}$ and which has the same exponent as σ.

Proposition 4. *Let a uni-monotonous state σ and $\mathbf{A} \in \mathcal{g}$ be given. Then $\iota_{\mathbf{A}}(\sigma)$ is again a uni-monotonous state.*

Proof: Denote the exponent of σ by $\mathbf{E}$ and its configuration by $\mathbf{G}$. By Prop.3, $\iota_{\mathbf{A}}(\sigma)$ is a stable monotonous state. Its exponent is $\mathbf{AEA}^{-1}$ and, by (8.8) of [N3], its configuration is $(\mathbf{A}^{-1})^\top \mathbf{GA}^{-1}$. Now let τ be any stable monotonous state whose exponent is $\mathbf{AEA}^{-1}$ and whose configuration is $(\mathbf{A}^{-1})^\top \mathbf{GA}^{-1}$. Applying Prop.3 to the case when σ is replaced by τ and $\mathbf{A}$ by $\mathbf{A}^{-1}$, we see that $\iota_{\mathbf{A}^{-1}}(\tau)$ is a stable monotonous state whose exponent is $\mathbf{E}$ and whose configuration is $\mathbf{G}$. By the uniqueness of σ, we have $\iota_{\mathbf{A}^{-1}}(\tau) = \sigma$ and hence $\iota_{\mathbf{A}}(\sigma) = \tau$. ∎

Recall that the orbits in the state space Σ under the action of the symmetry group $\mathcal{g}$ are called **reduced states** (see Sect.10 of [N7]) and that the orbit of a given state σ is defined by

$$\Omega_\sigma := \{\iota_{\mathbf{A}}(\sigma) \mid \mathbf{A} \in \mathcal{g}\} \ . \tag{3.1}$$

We see that Prop.4 can be phrased in the following way: If a reduced state contains one uni-monotonous state, then it consists entirely of uni-monotonous states. Such a reduced state will be called a **uni-monotonous reduced state**.

We assume now that a uni-monotonous state σ is given and we denote its exponent by $\mathbf{E}$ and its configuration by $\mathbf{G}$.

Proposition 5. *The symmetry group $\mathcal{g}_\sigma$ of σ, as defined by (8.11) of [3], satisfies*

$$\mathcal{g} \ \cap \ \mathrm{Orth}\,\mathbf{G} \ \cap \ \mathrm{Com}\,\mathbf{E} \quad \subset \quad \mathcal{g}_\sigma \quad \subset \quad \mathcal{g} \ \cap \ \mathrm{Orth}\,\mathbf{G} \ . \tag{3.2}$$

Proof: Let $\mathbf{A} \in \ \mathcal{g} \cap \mathrm{Orth}\,\mathbf{G} \cap \mathrm{Com}\,\mathbf{E}$ be given. By (1.7) we have $\mathbf{AEA}^{-1} = \mathbf{E}$, and by (6.2) and (8.5)in [N3] we have $\mathbf{G} = (\mathbf{A}^{-1})^\top \mathbf{GA}^{-1} = \hat{\mathbf{G}}(\iota_{\mathbf{A}}(\sigma))$. Therefore, by the uniqueness of σ, we must have $\iota_{\mathbf{A}}(\sigma) = \sigma$, i.e. $\mathbf{A} \in \mathcal{g}_\sigma$. We conclude that the first inclusion of (3.2) is valid; the second is merely a restatement of (8.12) in [3]. ∎

The following is an immediate consequence of Prop.5 and of (8.9) and (8.11) in [3].

Proposition 6. *The intrinsic stress* $\mathbf{S} := \hat{\mathbf{S}}(\sigma)$ *produced by the given uni-monotonous state* σ *satisfies*

$$\mathbf{A}\mathbf{S}\mathbf{A}^{\top} = \mathbf{S} \quad \text{for all} \quad \mathbf{A} \in g \cap \operatorname{Orth}\mathbf{G} \cap \operatorname{Com}\mathbf{E} \tag{3.3}$$

We recall (see the end of Sect.17 of [N7]) that a monotonous process is called a **simple shearing** if its exponent $\mathbf{E}$ satisfies $\mathbf{E}^2 = \mathbf{0}$ but is not zero. The following fact of linear algebra will be needed.

Proposition 7. *We have* $\mathbf{E}^2 = \mathbf{0}$ *but* $\mathbf{E} \neq \mathbf{0}$ *if and only if there is a* $\mathbf{G}$*-orthonormal basis* $\mathbf{b}$ *of* $\mathcal{T}$ *such that the matirx* $[\mathbf{E}]_{\mathbf{b}}$ *of* $\mathbf{E}$ *relative to* $\mathbf{b}$ *has the form*

$$[\mathbf{E}]_{\mathbf{b}} := \kappa \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad \text{where} \quad \kappa \in \mathbb{P}^{\times} \ . \tag{3.4}$$

Proof: If (3.4) holds it is evident that $\mathbf{E}^2 = \mathbf{0}$ but $\mathbf{E} \neq \mathbf{0}$. Assume, conversely, that $\mathbf{E}^2 = \mathbf{0}$ but $\mathbf{E} \neq \mathbf{0}$. We then have $\operatorname{Rng}\mathbf{E} \subset \operatorname{Null}\mathbf{E}$ and hence $\dim\operatorname{Rng}\mathbf{E} \leq \dim\operatorname{Null}\mathbf{E}$. This is compatible with the equality $\dim\operatorname{Rng}\mathbf{E} + \dim\operatorname{Null}\mathbf{E} = 3$ (see the Theorem on Dimension of Range and Nullspace in Sect.7 of [FDS]) and with $\mathbf{E} \neq \mathbf{0}$ only if $\dim\operatorname{Rng}\mathbf{E} = 1$ and $\dim\operatorname{Null}\mathbf{E} = 2$. Now choose a $\mathbf{G}$-unit vector $\mathbf{b}_1$ in the $\mathbf{G}$-orthogonal supplement of $\operatorname{Null}\mathbf{E}$. Since $\mathbf{E} \neq \mathbf{0}$ we must have $\mathbf{E}\mathbf{b}_1 \neq \mathbf{0}$ and hence we can determine $\kappa \in \mathbb{P}^{\times}$ and a $\mathbf{G}$-unit vector $\mathbf{b}_2$ such that $\mathbf{E}\mathbf{b}_1 = \kappa\mathbf{b}_2$. Choosing a $\mathbf{G}$-unit vector $\mathbf{b}_3$ that is $\mathbf{G}$-orthogonal to both $\mathbf{b}_1$ and $\mathbf{b}_2$ we obtain a $\mathbf{G}$-orthogonal basis $\mathbf{b} := (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ with the desired property (3.4). ∎

Definition 5. *We say that a uni-monotonous state is a* **state of simple shearing** *if its exponent* $\mathbf{E}$ *satisfies* $\mathbf{E}^2 = \mathbf{0}$.

Proposition 8. *Assume that* σ *is a state of simple shearing and let* $\mathbf{S} := \hat{\mathbf{S}}(\sigma)$ *be the intrinsic stress produced by* σ*. Determine, according to Prop.7, a* $\mathbf{G}$*-orthonormal basis* $\mathbf{b}$ *of* $\mathcal{T}$ *such that the matrix of* $\mathbf{E}$ *relative to* $\mathbf{b}$ *has the form (3.4). Assume, also, that the lineon* $\mathbf{A} \in \operatorname{Lin}\mathcal{T}$ *whose matrix relative to* $\mathbf{b}$ *is*

$$[\mathbf{A}]_{\mathbf{b}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.5}$$

belongs to the symmetry group g*. Then the matrix* T *of* $\mathbf{S}\mathbf{G} \in \operatorname{Lin}\mathcal{T}$ *relative to* $\mathbf{b}$ *is symmetric and has the form*

$$[\mathbf{S}\mathbf{G}]_{\mathbf{b}} =: T = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \ . \tag{3.6}$$

Proof: The symmetry of T follows from the symmetry of $\mathbf{S}$. It is clear that $\mathbf{A} \in \operatorname{Orth}\mathbf{G} \cap \operatorname{Com}\mathbf{E}$. In view of (6.2) in [3], we see that $\mathbf{A}^{\top}\mathbf{G} = \mathbf{G}\mathbf{A}^{-1}$.

Therefore, since $\mathbf{A} \in g$, it follows from Prop.6 that $\mathbf{SG} = \mathbf{ASA}^\top\mathbf{G} = \mathbf{ASGA}^{-1}$ and hence that

$$T = [\mathbf{A}]_\mathbf{b} T\, [\mathbf{A}]_\mathbf{b}{}^{-1} \ . \tag{3.7}$$

By the rules of matrix multiplication, it follows from (3.5) that

$$[\mathbf{A}]_\mathbf{b} T\, [\mathbf{A}]_\mathbf{b}{}^{-1} = \begin{bmatrix} T_{11} & T_{12} & -T_{13} \\ T_{12} & T_{22} & -T_{23} \\ -T_{13} & -T_{23} & T_{33} \end{bmatrix} \ ,$$

which is compatible with (3.7) only if T is of the form (3.6). ∎

4. Semi-liquid material elements.

Now we assume that a *semi-liquid* simple material element, in the sense of Def.1 in Sect.8 of [N3], is given.

Proposition 9. *Given any* $\mathbf{G}_0$ *in the set* $\mathcal{G}$ *of all configurations of the element, we have*

$$\mathcal{G} = \{\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*) \mid \det(\mathbf{G}_0{}^{-1}\mathbf{G}) = 1\} \ = \ \{\mathbf{A}^\top\mathbf{G}_0\mathbf{A} \mid \mathbf{A} \in g\} \ . \tag{4.1}$$

The proof is exactly the same as that of Prop.19.1 of [N7] and will not be repeated here.

Because of the incompressibility of the element, we must now interpret $\hat{\mathbf{S}}(\sigma)$ to be the intrinsic **extra-stress** produced by a given $\sigma \in \Sigma$, which differs from the intrinsic stress by an indeterminate pressure-term of the form $-p(\hat{\mathbf{G}}(\sigma))^{-1}$ with $p \in \mathbb{R}$. It is useful to fix $\hat{\mathbf{S}}$ by the normalization condition

$$\mathrm{tr}\,(\hat{\mathbf{S}}(\sigma)\hat{\mathbf{G}}(\sigma)) = 0 \qquad \text{for all} \qquad \sigma \in \Sigma \ . \tag{4.2}$$

(See Sect.8 of [N7].)

Using the differentiation rule (0.1) for the determinant function it follows from (1.3) and the Chain Rule that the function $\mathbf{F}$ given by (1.2) satisfies

$$(\det \circ \mathbf{F})^\bullet = (\det \circ \mathbf{F})\mathrm{tr}\,\mathbf{E} \ , \quad (\det \circ \mathbf{F})(0) = 1 \ . \tag{4.3}$$

As in [N7] we use the notation

$$\mathrm{Lin}_0\mathcal{T} := \{\mathbf{L} \in \mathrm{Lin}\,\mathcal{T} \mid \mathrm{tr}\,\mathbf{L} = 0\} \tag{4.4}$$

for the set of all traceless lineons on $\mathcal{T}$. The following result is an immediate consequence of (4.3), (1.2) and the definition of $g = \mathrm{Unim}\,\mathcal{T}$ given by (6.3) of [3].

Proposition 10. *For every* $\mathbf{E} \in \mathrm{Lin}_0\mathcal{T}$, *we have* $\exp(t\mathbf{E}) \in g$ *for all* $t \in I\!R$, *i.e. the condition (2.1) is satisfied. Hence every* $\mathbf{E} \in \mathrm{Lin}_0\mathcal{T}$ *is a candidate for the exponent of a monotonous state.*

Proposition 11. *Let a uni-monotonous state* σ_0 *be given. Denote its exponent by* $\mathbf{E}_0$ *and its configuration by* $\mathbf{G}_0$. *Let a* $\mathbf{G}_0$*-orthonormal basis* $\mathbf{b}^0$, *a configuration* $\mathbf{G} \in \mathcal{G}$, *and a* $\mathbf{G}$*-orthonormal basis* $\mathbf{b}$ *be given. Denote by* $\mathbf{E} \in \mathrm{Lin}\,\mathcal{T}$ *the lineon whose matrix relative to* $\mathbf{b}$ *is the same as the matrix of* $\mathbf{E}_0$ *relative* $\mathbf{b}^0$, *so that*

$$[\mathbf{E}_0]_{\mathbf{b}^0} = [\mathbf{E}]_{\mathbf{b}}. \tag{4.5}$$

Then there is exactly one uni-monotonous state σ *whose configuration is* $\mathbf{G}$ *and whose exponent is* $\mathbf{E}$. *The states* σ *and* σ_0 *belong to the same reduced uni-monotonous state, i. e. we have*

$$\sigma = \iota_{\mathbf{A}}(\sigma_0) \quad \text{for some} \quad \mathbf{A} \in g\ . \tag{4.6}$$

Finally, putting $\mathbf{S}_0 := \hat{\mathbf{S}}(\sigma_0)$ *and* $\mathbf{S} := \hat{\mathbf{S}}(\sigma)$, *the matrix of* $\mathbf{GS}$ *relative to* $\mathbf{b}$ *is the same as the matrix of* $\mathbf{S}_0\mathbf{G}_0$ *relative to* $\mathbf{b}^0$, *i. e. we have*

$$[\mathbf{S}_0\mathbf{G}_0]_{\mathbf{b}^0} = [\mathbf{SG}]_{\mathbf{b}}. \tag{4.7}$$

Proof: We denote by $\mathbf{A} \in \mathrm{Lis}\,\mathcal{T}$ the lineon which sends the basis $\mathbf{b}^0$ to the basis $\mathbf{b}$, so that

$$\mathbf{A}{\mathbf{b}^0}_i = \mathbf{b}_i \qquad \text{for all} \qquad i \in \{1,2,3\}. \tag{4.8}$$

It then follows from (4.5) that

$$\mathbf{E} = \mathbf{A}\mathbf{E}_0\mathbf{A}^{-1}\ . \tag{4.9}$$

To say that $\mathbf{b}^0$ is $\mathbf{G}_0$-orthonormal and that $\mathbf{b}$ is $\mathbf{G}$-orthonormal means, by (6.1) in [N3], that

$$(\mathbf{G}_0{\mathbf{b}^0}_i){\mathbf{b}^0}_j = \delta_{i,j} = (\mathbf{G}\mathbf{b}_i)\mathbf{b}_j \qquad \text{for all} \quad i,j \in \{1,2,3\}\ . \tag{4.10}$$

Substituting (4.8) into (4.10), we see that

$$\begin{aligned}(\mathbf{G}_0{\mathbf{b}^0}_i){\mathbf{b}^0}_j &= (\mathbf{G}\mathbf{b}_i)\mathbf{b}_j = (\mathbf{G}\mathbf{A}{\mathbf{b}^0}_i)\mathbf{A}{\mathbf{b}^0}_j \\ &= ((\mathbf{A}^\top\mathbf{G}\mathbf{A}){\mathbf{b}^0}_i){\mathbf{b}^0}_j \quad \text{for all} \quad i,j \in \{1,2,3\}\ ,\end{aligned}$$

which implies

$$\mathbf{A}^\top\mathbf{G}\mathbf{A} = \mathbf{G}_0\ . \tag{4.11}$$

Using Prop.9, (4.11), and basic properties of determinants we see that

$$1 = \det(\mathbf{G}^{-1}\mathbf{G}_0) = \det(\mathbf{G}^{-1}\mathbf{A}^\top\mathbf{G})\det\mathbf{A} = \det(\mathbf{A}^\top)\det\mathbf{A} = (\det\mathbf{A})^2 \tag{4.12}$$

and hence that $|\det \mathbf{A}| = 1$, which shows that $\mathbf{A} \in g$. It follows from Prop.4, from (4.9), from Prop.3, from (4.11), and from (8.8) of [3] that $\sigma := \iota_{\mathbf{A}}(\sigma_0)$ has the desired properties.

The condition (8.9) of [3] implies that $\mathbf{S} = \mathbf{A}\mathbf{S}_0\mathbf{A}^\top$. Using (4.11), it follows that

$$\mathbf{S}\mathbf{G} = \mathbf{A}\mathbf{S}_0\mathbf{A}^\top\mathbf{G} = \mathbf{A}\mathbf{S}_0\mathbf{A}^\top\mathbf{G}\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}(\mathbf{S}_0\mathbf{G}_0)\mathbf{A}^{-1} \ ,$$

which, in view of (4.8), shows that (4.7) is valid. ∎

Prop.11 shows that a given reduced uni-monotonous state Ω can be characterized by a matrix $B \in \mathrm{Lin}_0\mathbb{R}^3$ in the following way: Given any configuration $\mathbf{G} \in \mathcal{G}$ and any $\mathbf{E} \in \mathrm{Lin}_0\mathcal{T}$ whose matrix relative to some $\mathbf{G}$-orthonormal basis is B, there is exactly one $\omega \in \Omega$ whose configuration is $\mathbf{G}$ and whose exponent is $\mathbf{E}$. Given any orthogonal matrix $R \in \mathrm{Orth}\,\mathbb{R}_3$, the matrices B and $RBR^\top$ determine the same reduced uni-monotonous state. One cannot assert that *every* matrix $B \in \mathrm{Lin}_0\mathbb{R}^3$ characterizes a reduced uni-monotonous state. However, from now on we will make the following assumption:

Assumption I. *For every $\kappa \in \mathbb{P}^\times$ the matrix*

$$B = \kappa U \qquad \text{with} \quad U := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \kappa \in \mathbb{P}^\times \ , \tag{4.13}$$

does characterize a reduced uni-monotonous state Ω_κ.

It follows from Prop.7 that, for every $\kappa \in \mathbb{P}^\times$, the members of Ω_κ are all states of simple shearing. Hence Prop.8 yields:

Proposition 12. *One can determine three functions*

$$\tau \ , \ \sigma_1 \ , \ \sigma_2 \ : \mathbb{P}^\times \longrightarrow \mathbb{R} \tag{4.14}$$

with the following property: Let $\kappa \in \mathbb{P}^\times$ be given. Given any configuration $\mathbf{G}$ and any $\mathbf{G}$-orthonormal basis $\mathbf{b}$, let $\omega_\kappa \in \Omega$ be the state of simple shearing determined by $\mathbf{G}$ and $\mathbf{b}$ according to Assumption I and Prop.11. Denote by $\mathbf{S}_\kappa$ the intrinsic extra-stress produced by ω_κ. Then the matrix T of $\mathbf{S}_\kappa\mathbf{G}$ relative to $\mathbf{b}$ has the form

$$[\mathbf{S}_\kappa\mathbf{G}]_{\mathbf{b}} =: T = \begin{bmatrix} T_{11} & \tau(\kappa) & 0 \\ \tau(\kappa) & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \tag{4.15}$$

with

$$\sigma_1(\kappa) = T_{11} - T_{33} \ , \quad \sigma_2(\kappa) = T_{22} - T_{33} \ . \tag{4.16}$$

The functions τ , σ_1 , σ_2 are called the **visco-metric functions** of the given semi-liquid element. Specifically, τ is called the **shear stress function**, and σ_1 and σ_2 are called the **normal stress functions**.

Since $\mathrm{tr}T = T_{11} + T_{22} + T_{33} = 0$ by the normalization (4.2), it follows from (4.15) and (4.16) that the matrix T is completely determined by the values at κ of the viscometric functions. If we replace in (4.15) the intrinsic extra-stress $\mathbf{S}_\kappa$ by the intrinsic stress, which is of the form $\mathbf{S}_\kappa - p\,(\hat{\mathbf{G}}(\sigma))^{-1}$, then the formulas (4.15) and (4.16) remain valid but the trace of T need no longer be zero.

Assumption II. *Let a configuration* $\mathbf{G}$ *and a* $\mathbf{G}$*-orthonormal basis* $\mathbf{b}$ *be given. For every* $\kappa \in \mathbb{P}^\times$*, let* $\omega_\kappa \in \Omega$ *be the state of simple shearing determined as in Prop.12. Then* ω_κ *depends continuously on* κ *and has a limit* $\omega_0 := \lim_{\kappa\to 0}\omega_\kappa$.

In view of the continuity of the function $\hat{\mathbf{S}}$, it follows that $\mathbf{S}_\kappa := \hat{\mathbf{S}}(\omega_\kappa)$ depends continuously on κ and has a limit

$$\mathbf{S}_0 := \lim_{\kappa\to 0}\mathbf{S}_\kappa \ . \tag{4.17}$$

Hence, by (4.15) and (4.16), the viscometric functions (4.14) are continuous and have limits

$$\tau_0 := \lim_{\kappa\to 0}\tau(\kappa)\ ,\quad \sigma_{10} := \lim_{\kappa\to 0}\sigma_1(\kappa)\ ,\quad \sigma_{20} := \lim_{\kappa\to 0}\sigma_2(\kappa)\ . \tag{4.18}$$

Remark 5. It is clear that the limit ω_0 in Assumption II is a monotonous state of exponent $\mathbf{0}$, i.e. a relaxed state, and that its configuration is $\mathbf{G}$. The state ω_0 need not be a uni-monotonous state because it may depend on the choice of the $\mathbf{G}$-orthonormal basis $\mathbf{b}$. Therefore, there is no reason for ω_0 to be isotropic and hence for the limits (4.18) to be zero. However, if the element is semi-elastic and hence an element of a simple liquid in the sense of Def.1 in Sect.8 of [3], then ω_0 is the *only* relaxed state whose configuration is $\mathbf{G}$. In this case, ω_0 is uni-monotonous with exponent zero and hence isotropic by Prop.5. Then we have $\mathbf{S}_0 = \mathbf{0}$ and the limits in (4.18) are all zero. The viscometric functions then reduce to the ones considered, for example, in Sect.10 of [CMN]. ∎

5. Monotonous and viscometric flows.

We now assume that a continuous body system $\mathcal{B}$ and a motion $\mu : \mathcal{B}\times I \longrightarrow \mathcal{F}$ are given as described in Sect.3 of [3]. We also use the notations of Sect.3 of [3]. We assume that the motion is of class C^2.

Let $t \in I$ be given. The **relative transplacement** function $\chi_t : \mathcal{R}_t \times I \longrightarrow \mathcal{F}$ is defined by

$$\chi_t(\mathbf{x}, r) := \mu(\mu_t{}^{\leftarrow}(\mathbf{x}), r) \qquad \text{for all} \quad r \in I\ ,\ \mathbf{x} \in \mathcal{R}_t\ . \tag{5.1}$$

We denote its gradient by

$$\mathbf{F}_t(\mathbf{x}, r) := \nabla_{\mathbf{x}}\chi_t(\cdot, r) \qquad \text{for all} \quad r \in I\ ,\ \mathbf{x} \in \mathcal{R}_t\ . \tag{5.2}$$

Putting

$$\mathcal{W} := \{(\mathbf{z}, t) \mid t \in I\ ,\ \mathbf{z} \in \mathcal{R}_t\}\ , \tag{5.3}$$

we define the **velocity field** $\mathbf{v} : \mathcal{W} \longrightarrow \mathcal{V}$ of the given motion by

$$\mathbf{v}(\mathbf{z}, t) := (\chi_t({\chi_t}^{\leftarrow}(\mathbf{z}), \cdot))^{\bullet}(t) \qquad \text{for all} \quad (\mathbf{z}, t) \in \mathcal{W} \ , \tag{5.4}$$

so that

$$\mathbf{v}(\chi_t(X), t) := (\chi_t(X, \cdot))^{\bullet}(t) \qquad \text{for all} \quad X \in \mathcal{B} \ , \ t \in I \ . \tag{5.5}$$

Often, a motion is characterized by the prescription of a set $\mathcal{W}$ of the form (5.5) and a velocity field $\mathbf{v} : \mathcal{W} \longrightarrow \mathcal{V}$. Given $t \in I$, the relative transplacement function χ_t can then be obtained by noting that, for each $x \in \mathcal{R}_t$, the function $\chi_t(\mathbf{x}, \cdot) : \mathcal{R}_t \longrightarrow \mathcal{E}$ is the solution of the initial-value problem described by

$$(\chi_t(\mathbf{x}, \cdot))^{\bullet} = \mathbf{v}(\chi_t(\mathbf{x}, \cdot), \cdot) \ \ , \ \ \chi_t(\mathbf{x}, t) = \mathbf{x} \ . \tag{5.6}$$

We now fix $X \in \mathcal{B}$ and consider the motion $\mathbf{M} : I \longrightarrow \mathrm{Lis}\,(\mathcal{T}_X, \mathcal{V})$ of the element $\mathcal{T}_X$ induced by the motion μ as defined by (4.3) in [N3]. We also fix $t \in I$ and put $\mathbf{x} := \chi_t(X)$. In view of (5.1), (5.2), and (4.3) of Sect.4 in [N3] we have

$$\mathbf{F}_t(\mathbf{x}, r) = \mathbf{M}(r)\mathbf{M}(t)^{-1} \qquad \text{for all} \quad r \in I \ . \tag{5.7}$$

Now, $I_t := \mathbb{P} \cap (I - t)$ is a closed interval whose left endpoint is 0, and the mapping $\mathbf{P}_t : I_t \longrightarrow \mathrm{Pos}^{+}(\mathcal{T}_X, {\mathcal{T}_X}^{*})$, defined by

$$\mathbf{P}_t(s) := \mathbf{M}(t + s)^{\top}\mathbf{M}(t + s) \qquad \text{for all} \quad s \in I_t \ , \tag{5.8}$$

is a deformation process of the element $\mathcal{T}_X$. Given any $r \in I$ with $r \geq t$, we have

$$\mathbf{P}_t(s) = \mathbf{P}_r((r - t) + s) \qquad \text{for all} \quad s \in I_t \ . \tag{5.9}$$

If all the processes $\mathbf{P}_t$, with arbitrary $t \in I$, are monotonous, it is clear from (1.5) and (5.9) that they must all have the the same exponent.

Definition 6. *We say that the given motion* $\mathbf{M}$ *of* $\mathcal{T}_X$ *is a* **monotonous flow** *of exponent* $\mathbf{E} \in \mathrm{Lin}\,\mathcal{T}_X$ *if, for every* $t \in I$*, the deformation process defined by (5.8) is a monotonous process of exponent* $\mathbf{E}$*. In the case when* $\mathbf{E}^2 = \mathbf{0}$*, we say that* $\mathbf{M}$ *is a* **viscometric flow***.*

The proof of the following result is almost the same as that of Prop.17.3 of [N7] and will not be repeated here.

Proposition 13. *The given motion* $\mathbf{M}$ *of* $\mathcal{T}_X$ *is a monotonous flow if and only if, for every* $t \in I$*, there is a* $\mathbf{B} \in \mathrm{Lin}\,\mathcal{V}$ *and a mapping* $\mathbf{R} : I \longrightarrow \mathrm{Orth}\,\mathcal{V}$ *such that*

$$\mathbf{F}_t(\mathbf{x}, r) = \mathbf{R}(r)\exp(r\mathbf{B}) \qquad \text{for all} \quad r \in I \ , \tag{5.10}$$

where $\mathbf{x} = \mu_t(X)$*. The exponent of the flow is then given by*

$$\mathbf{E} = \mathbf{M}(t)^{-1}\mathbf{B}\mathbf{M}(t) \in \mathrm{Lin}\,\mathcal{T}_X \ . \tag{5.11}$$

In the case when $\mathbf{M}$ is a viscometric flow, we have $\mathbf{B}^2 = \mathbf{0}$ in (5.10) and hence, since $\exp(r\mathbf{B}) = \mathbf{1}_{\mathcal{V}} + r\mathbf{B}$ in this case, (5.10) reduces to

$$\mathbf{F}_t(\mathbf{x}, r) = \mathbf{R}(r)(\mathbf{1}_{\mathcal{V}} + r\mathbf{B}) \qquad \text{for all} \quad r \in I \ . \tag{5.12}$$

Proposition 14. *Assume that, for a given $\kappa \in \mathbb{P}^{\times}$, one can associate, with each $r \in I$, an orthonormal basis $\mathbf{e}(r)$ such that*

$$(\mathbf{e}(r)_i \mathbf{F}_t(x, r)\mathbf{e}(t)_j \mid i, j \in \{1, 2, 3\}) = \begin{bmatrix} 1 & 0 & 0 \\ r\kappa & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{for all} \quad r \in I \ . \tag{5.13}$$

Then $\mathbf{M}$ is a viscometric flow. Moreover, putting

$$\mathbf{G} := \mathbf{M}(t)^{\top}\mathbf{M}(t) \ \text{ and } \ \mathbf{b}_i := \mathbf{M}(t)\mathbf{e}(t)_i \qquad \text{for all} \quad i \in \{1, 2, 3\} \ , \tag{5.14}$$

$\mathbf{b}$ is a $\mathbf{G}$-orthonormal basis and the matrix $B := [\mathbf{E}]_{\mathbf{b}}$ of the exponent $\mathbf{E}$ of $\mathbf{M}$ relative to $\mathbf{b}$ has the form (4.13).

Proof: It follows from (5.13) that (5.12) is satisfied when $\mathbf{B} \in \operatorname{Lin}\mathcal{V}$ is the lineon whose matrix $B := [\mathbf{B}]_{\mathbf{e}(t)}$ relative to the basis $\mathbf{e}(t)$ has the form (4.13) and when $\mathbf{R} : I \longrightarrow \operatorname{Orth}\mathcal{V}$ is determined by the condition that $\mathbf{R}(r)\mathbf{e}(t)_i = \mathbf{e}(r)_i$ for all $i \in \{1, 2, 3\}$. Hence Prop.13 shows that $\mathbf{M}$ is viscometric. The assertion concerning the matrix of $\mathbf{E}$ follows from (5.11). ∎

We now go back to the given motion μ of the whole body $\mathcal{B}$. We say that this motion is a **monotonous flow [viscometric flow]** if, for *every* $X \in \mathcal{B}$, the motion $\mathbf{M}$ of the element $\mathcal{T}_X$ induced by μ according to (5.3) of [3] is a monotonous flow [viscometric flow]. Important examples of viscometric flows are the *curvilineal flows* considered in Sect.18 of [CMN]. In fact, it is Prop.14 above that is used there to show that the curvilineal flows are viscometric.

6. Special viscometric flows of semi-liquids.

In this section, we assume that a materially uniform semi-liquid body system $\mathcal{B}$ is given. (See [3], end of Sect.4, for definition of *material uniform* and Def.1 of Sect.8 for *semi-liquid*.) This means that all the material elements of $\mathcal{B}$ are semi-liquid and materially isomorphic to each other. We assume the validity of Assumptions I and II in Sect.4 . It follows that the viscometric functions described in Prop.12 are the same for all the elements of $\mathcal{B}$.

We now consider a viscometric flow $\mu : I \times \mathcal{B} \longrightarrow \mathcal{F}$ of $\mathcal{B}$ in a given frame space $\mathcal{F}$ as defined in the previous section. Assume, for a moment, that the $I := \mathbb{R}$. Physically, this means that the flow takes place for all time, past as well as future. Although this may be unrealistic, it can be used as an idealization of a flow that got started sufficiently long ago and has no definite end. Making use of Assumption I stated in Sect.4, we can then expect, by Def.3 of Sect.4, that each of the material elements of the body has reached a state of simple

shearing as defined by Def.5. From now on we will assume that this is the case, even though we leave open the specific nature of the interval I. It follows that the results of Sect.4 apply.

Let $X \in \mathcal{B}$ be given and consider the viscometric flow $\mathbf{M}$ of the element $\mathcal{T}_X$ induced by μ as defined by (4.3) in [3]. Let $t \in I$ be fixed. Then the configuration of the element at time t is $\mathbf{G} := \mathbf{P}(t) = \mathbf{M}(t)^\top \mathbf{M}(t)$ (see (4.4) in [3]). Denote the state of simple shearing of the element at time t by ω. The (Cauchy-) stress $\mathbf{T} \in \mathrm{Sym}\mathcal{V}$ is related to the intrinsic stress $\mathbf{S} := \hat{\mathbf{S}}(\omega)$ by

$$\mathbf{T} = \mathbf{M}(t)\mathbf{S}\mathbf{M}(t)^\top = \mathbf{M}(t)\mathbf{S}\mathbf{G}\mathbf{M}(t)^{-1} \ . \tag{6.1}$$

(See (6.1) in [3].)

Now let $\mathbf{B} \in \mathrm{Lin}\,\mathcal{V}$ be determined according to Prop.13, so that (5.11) holds when $\mathbf{E} \in \mathrm{Lin}\,\mathcal{T}_X$ is the exponent of the flow $\mathbf{M}$. Since $\mathbf{B}^2 = \mathbf{0}$ one can choose, by Prop.7, a basis $\mathbf{e}(t)$ such that the matrix $B := [\mathbf{B}]_{\mathbf{e}(t)}$ of $\mathbf{B}$ relative to $\mathbf{e}(t)$ has the form (4.13). Now, define the $\mathbf{G}$-orthonormal basis $\mathbf{b}$ of $\mathcal{T}_X$ by $(5.14)_2$, so that

$$B = [\mathbf{E}]_{\mathbf{b}} = [\mathbf{B}]_{\mathbf{e}(t)} = \begin{bmatrix} 0 & 0 & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with} \ \ \kappa \in \mathbb{P}^\times \ , \tag{6.2}$$

It follows from (6.1) and (5.14) that $[\mathbf{T}]_{\mathbf{b}} = [\mathbf{S}\mathbf{G}]_{\mathbf{e}(t)}$ and hence, by Prop.12, that the matrix T of the stress $\mathbf{T}$ relative to $\mathbf{e}(t)$ is related to the viscometric functions of Prop.12 by

$$[\mathbf{T}]_{\mathbf{e}(t)} =: T = \begin{bmatrix} T_{11} & \tau(\kappa) & 0 \\ \tau(\kappa) & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \tag{6.3}$$

with

$$\sigma_1(\kappa) = T_{11} - T_{33} \ , \quad \sigma_2(\kappa) = T_{22} - T_{33} \ . \tag{6.4}$$

We conclude that all the considerations of Chapters III and IV of [CMN] can be applied here, provided they are modified to account for the possibility that the limits (4.18) need not be zero. Before investigating some of these modifications we add the following to the Assumptions I and II stated in Sect.4.

Assumption III. *The shear stress function τ of (6.14) is strictly isotone and we have*

$$\tau_0 := \lim_{\kappa \to 0} \tau(\kappa) > 0, \quad \lim_{\kappa \to \infty} \tau(\kappa) = \infty \ . \tag{6.5}$$

This assumption is analogous to the ones mentioned at the end of Sect.11 of [CMN]. It insures that $\mathrm{Rng}\,\tau = \tau_0 + \mathbb{P}^\times$ and that $\tau|^{\mathrm{Rng}}$ is invertible. The **rate of shear function** $\lambda : \mathbb{P} \longrightarrow \mathbb{P}$ introduced in Sect.11 of [CMN] must be given the following new definition:

$$\lambda(S) := \begin{cases} 0 & \text{if } S \in [0, \tau_0] \\ (\tau|^{\mathrm{Rng}})^{\leftarrow}(S) & \text{if } S \in \tau_0 + \mathbb{P}^\times \end{cases} \ , \tag{6.6}$$

so that

$$\lambda(\tau(\kappa)) = \kappa \quad \text{for all} \quad \kappa \in \mathbb{P}^\times, \quad \lambda(S) := 0 \quad \text{for all} \quad S \in [0, \tau_0] \ . \tag{6.7}$$

It is this function that must be used when dealing with semi-liquids.

We now consider two special viscometric flows.

A. Poiseuille Flow.

The formulas of Sect.19 of [CMN] remain valid with the definition of λ given by (6.6). In these formulas, R is the radius of the cylindrical tube through which the flow takes place, f is the *driving force*, i.e. the force per unit volume that produces the flow, and $u : [0, R] \longrightarrow \mathbb{P}$ describes the axial speed of the flow as a function of the distance from the axis of the tube. The formula (19.7) of [CMN] reads

$$u(r) = \int_r^R \lambda(\frac{uf}{2})\mathrm{d}u \quad \text{for all} \quad r \in [0, R] \ . \tag{6.8}$$

We put

$$f_0 := \frac{2\tau_0}{R} \quad \text{and} \quad R_p := \frac{2\tau_0}{f} \ . \tag{6.9}$$

If $f \leq f_0$ and hence $R_p \geq R$, then (6.8) and (6.7) give $u(r) = 0$ for all $r \in [0, R]$, showing that the material cannot move at all.

We now assume that $f > f_0$ and hence $R_p < R$, and we put

$$u_p := \int_{R_p}^R \lambda(\frac{uf}{2})\mathrm{d}u \ . \tag{6.10}$$

We then have $u(r) = u_p$ for all $r \in [0, R_p]$, showing that the material occupying the cylinder of radius R_p moves rigidly at constant speed u_p. This cylindrical part of the body is called the **plug**, so that u_p is the speed of the plug. The formula (6.8) remains relevant when $R_p \leq r \leq R$.

Let $Q : f_0 + \mathbb{P}^\times \longrightarrow \mathbb{P}$ describe the *volume discharge*, i.e. the volume of material passing through the tube per unit time, as a function of the driving force that produces the flow. It follows from (19.9) of [CMN] and (6.7) above that

$$Q(f) = \pi \int_{R_p}^R r^2 \lambda(\frac{rf}{2})\mathrm{d}r = \frac{8\pi}{f^3} \int_{\tau_0}^{\frac{Rf}{2}} S^2 \lambda(S)\mathrm{d}S \quad \text{for all} \quad f \in f_0 + \mathbb{P}^\times \ . \tag{6.11}$$

This formula can be used to express the function λ in terms of the function Q. In fact, we have

$$\lambda(S) = \frac{1}{4\pi R S^2} \bar{Q}^\bullet(\frac{2S}{R}) \quad \text{for all} \quad S \in \tau_0 + \mathbb{P}^\times \ , \tag{6.12}$$

where $\bar{Q} : f_0 + \mathbb{P}^\times \longrightarrow \mathbb{P}$ is defined by

$$\bar{Q}(f) := f^3 Q(f) \quad \text{for all} \quad f \in f_0 + \mathbb{P}^\times \ . \tag{6.13}$$

The function Q may be amenable to determination by experiment. One can then use (6.12) and (6.13) to calculate the rate of shear function λ and hence the shear stress function τ.

B. Couette Flow.

Most of the formulas of Sect.17 of [CMN] remain valid with the definition of λ given by (6.6). In these formulas R_1 and R_2, with $R_1 < R_2$ denote the radii of the inner and outer cylinder between which the flow takes place, M is the torque per unit height that produces the flow, and $\omega : [R_1, R_2] \longrightarrow \mathbb{P}$ describes the angular speed of the flow as a function of the distance from the axis of the cylinders. The angular speeds of the two bounding cylinders are

$$\Omega_1 := \omega(R_1) \quad \text{and} \quad \Omega_2 := \omega(R_2) \ . \tag{6.14}$$

The formula (17.5) of [CMN] reads

$$\omega(r) - \Omega_1 = \int_{R_1}^{r} \frac{1}{u} \lambda(\frac{M}{2\pi u^2}) \mathrm{d}u \qquad \text{for all} \quad r \in [R_1, R_2] \ . \tag{6.15}$$

We put

$$M_1 := 2\pi {R_1}^2 \tau_0 \ , \quad M_2 := 2\pi {R_2}^2 \tau_0 \ , \quad R_p := \sqrt{\frac{M}{2\pi\tau_0}} \ . \tag{6.16}$$

If $M \leq M_1$ and hence $R_p \leq R_1$, then (6.7) shows that the integrand in (6.15) is zero and hence that $\omega(r) = \Omega_1 = \Omega_2$ for all $r \in [R_1, R_2]$, showing that the material rotates rigidly at constant angular speed. If $M \geq M_2$ and hence $R_p \geq R_2$, we have $\frac{M}{2\pi u^2} \in \tau_0 + \mathbb{P}$ for all $u \in [R_1, r]$ in (6.15), so that the integrand is nowhere zero and hence ω is strictly isotone. In this case, the results of Sect.17 of [CMN] apply without modification.

We now assume that $M_1 < M < M_2$ and hence that $R_1 < R_p < R_2$. Then (6.7), (6.14), and (6.15) show that

$$\Omega_2 - \Omega_1 = \int_{R_1}^{R_p} \frac{1}{u} \lambda(\frac{M}{2\pi u^2}) \mathrm{d}u \tag{6.17}$$

and that $\omega(r) = \Omega_2$ for all $r \in [R_p, R_2]$, which means that the the material occupying the region between the cylinder of radius R_p and the outer cylinder rotates rigidly at angular speed Ω_2. This part of the body is again called the **plug**, so that Ω_2 is the angular speed of the plug. The formula (6.15) remains relevant when $R_1 \leq r \leq R_p$.

Let $\Delta : [M_1, M_2] \longrightarrow \mathbb{P}$ describe the angular speed difference $\Omega_2 - \Omega_1$ as a function of the torque per unit height needed to produce the flow. Then (6.17) and (6.16) yield

$$\Delta(M) = \int_{\tau_0}^{\frac{M}{2\pi {R_1}^2}} \frac{\lambda(S)}{2S} \mathrm{d}S \qquad \text{for all} \qquad M \in [M_1, M_2] \ . \tag{6.18}$$

This formula can be used to express the function λ in terms of the function Δ. In fact, we have

$$\lambda(S) = 4\pi {R_1}^2 S \Delta^\bullet(2\pi {R_1}^2 S) \qquad \text{for all} \qquad S \in [\tau_0,\ (\frac{R_2}{R_1})^2 \tau_0] \ . \tag{6.19}$$

The function Δ may be amenable to determination by experiment. One can then use (6.19) to calculate the rate of shear function λ and hence the shear stress function τ. The result may then be compared with the one obtained from Poiseuille flow.

Remark 6. Assume that the shear stress function τ follows the simple rule

$$\tau(\kappa) = \tau_0 + \eta_0 \kappa \quad \text{for some} \quad \eta_0 \in \mathbb{P}^\times \quad \text{and} \quad \text{for all} \quad \kappa \in \mathbb{P}^\times \ . \tag{6.20}$$

Then, by (6.6), the rate of shear function λ satisfies

$$\lambda(S) = \frac{1}{\eta_0}(S - \tau_0) \qquad \text{for all} \qquad S \in \tau_0 + \mathbb{P} \tag{6.21}$$

and the integrals in the formulas (6.10), (6.11), and (6.17) can be evaluated explicitly. The results are

$$u_p = \frac{f}{4\eta_0}(R - R_p)^2 \ , \tag{6.22}$$

$$Q(f) = \frac{\pi R^4 f}{8\eta_0}\Big(1 - \frac{4}{3}(\frac{2\tau_0}{fR}) + \frac{1}{3}(\frac{2\tau_0}{fR})^4\Big) \qquad \text{for all} \qquad f \in f_0 + \mathbb{P}^\times \ , \tag{6.23}$$

and

$$\Omega_2 - \Omega_1 = \frac{\tau_0}{2\eta_0}\Big((\frac{R_p}{R_1})^2 - \log\big((\frac{R_p}{R_1})^2\big) - 1\Big) \ . \tag{6.24}$$

The formula (6.23) is known, in the literature of rheology, as the *Buckingham-Reiner* equation. ∎

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[5] Nematic Semi-Liquids

0. Introduction.

This paper is a sequel to [4]. To understand the mathematical background used here as well as in [4], the reader should be familiar with the concepts, terminology, and notation used in [3] and [FDS]. He should be comfortable, in particular, with 3-dimensional linear spaces that are *not* inner-product spaces. Given such a linear space $\mathcal{T}$, we apply the concepts familiar from inner product spaces to *each* of the members of $\mathrm{Pos}^+(\mathcal{T},\mathcal{T}^*)$ (see Sect.6 of [3]).

For later use, we record the following result from linear algebra.

Proposition 1. *Let* $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T},\mathcal{T}^*)$ *and* $\mathbf{N} \in \mathcal{T}$ *be given. Then* $\mathbf{N}$ *satisfies the three conditions*

$$\mathrm{tr}\mathbf{N} = 1, \quad \mathbf{N}^2 = \mathbf{N}, \quad \text{and} \quad \mathbf{GN} \in \mathrm{Sym}(\mathcal{T},\mathcal{T}^*) \tag{0.1}$$

if and only if there is $\mathbf{n} \in \mathcal{T}$ *such that*

$$\mathbf{N} = \mathbf{n} \otimes \mathbf{Gn} \quad \text{and} \quad (\mathbf{Gn})\mathbf{n} = 1\ . \tag{0.2}$$

The third of the conditions (0.1) says that $\mathbf{N}$ is symmetric relative to $\mathbf{G}$ and the second of the conditions (0.2) says that $\mathbf{n}$ is a $\mathbf{G}$-unit vector. Of course, if $\mathbf{n}$ is replaced by $-\mathbf{n}$, then (0.2) remains valid, and $\mathbf{n}$ is determined by the conditions (0.2) only to within a change of sign.

We use the following abbreviation:

$$\mathrm{Lin}_c\mathcal{T} := \{\mathbf{L} \in \mathrm{Lin}\,\mathcal{T} \mid \mathrm{tr}\,\mathbf{L} = c\} \text{ when } c = 0 \text{ or } 1. \tag{0.3}$$

For $c := 0$, (0.3) reduces to (4.4) of [N4].

1. The general model.

We consider here a specific model of a semi-liquid material element in the sense of Def.1 in Sect.6 of [3]. We describe the element with the same notation as is used in [3] and [4]. The underlying body element is described by the given linear space $\mathcal{T}$. The set $\mathcal{G}$ of possible configurations of the element is characterized by Prop.9 in Sect.4 of [4].

We assume that the state space Σ of the element consists of triples as follows:

$$\Sigma := \{(\mathbf{G},\mathbf{H},\mathbf{N}) \in \mathcal{G} \times \mathrm{Lin}_0\mathcal{T} \times \mathrm{Lin}_1\mathcal{T} \mid \mathbf{GH}, \mathbf{GN} \in \mathrm{Sym}(\mathcal{T},\mathcal{T}^*)\}\ . \tag{1.1}$$

The configuration of a given state $(\mathbf{G},\mathbf{H},\mathbf{N}) \in \Sigma$ is assumed to be specified by

$$\hat{\mathbf{G}}(\mathbf{G},\mathbf{H},\mathbf{N}) := \mathbf{G}. \tag{1.2}$$

As we shall see later in (1.11), the term $\mathbf{H}$ describes the rate of change of configuration. The term $\mathbf{N}$ can be interpreted to be the probability distribution of of alignments of rod-like molecules, as described in Sect.1.3.1 of [V]. For liquid crystals, $\mathbf{N}$ may be measurable by optical devices. In view of Prop.1, the condition $\mathbf{N}^2 = \mathbf{N}$ can be interpreted as stating that the rod-like molecules are all aligned in the same direction.

We assume that the function $\hat{\mathbf{S}}$ whose value at a given state gives the intrinsic stress produced by that state is determined by a continuous function

$$\hat{\mathbf{T}} : \mathrm{Lin}_0\mathcal{T} \ \times \ \mathrm{Lin}_1\mathcal{T} \longrightarrow \ \mathrm{Lin}\,\mathcal{T} \tag{1.3}$$

which satisfies, for all $\mathbf{H} \in \mathrm{Lin}_0\mathcal{T},\ \mathbf{N} \in \mathrm{Lin}_1\mathcal{T}$,

$$\hat{\mathbf{T}}(\mathbf{AHA}^{-1}, \mathbf{ANA}^{-1}) = \mathbf{A}\hat{\mathbf{T}}(\mathbf{H}, \mathbf{N})\mathbf{A}^{-1} \quad \text{for all} \quad \mathbf{A} \in \mathrm{Unim}\mathcal{T} \ , \tag{1.4}$$

and

$$\hat{\mathbf{T}}(\mathbf{G}^{-1}\mathbf{H}^\top\mathbf{G}, \mathbf{G}^{-1}\mathbf{N}^\top\mathbf{G}) = \mathbf{G}^{-1}\hat{\mathbf{T}}(\mathbf{H}, \mathbf{N})^\top\mathbf{G} \quad \text{for all} \quad \mathbf{G} \in \mathcal{G} \ , \tag{1.5}$$

Let $\sigma := (\mathbf{G}, \mathbf{H}, \mathbf{N}) \in \Sigma$ be given. It easily follows from (1.5) and (1.1) that $\hat{\mathbf{T}}(\mathbf{H}, \mathbf{N})\mathbf{G}^{-1} \in \mathrm{Sym}(\mathcal{T}^*, \mathcal{T})$ and hence, by (1.1), that it is meaningful to specify that the value of $\hat{\mathbf{S}}$ at σ be

$$\hat{\mathbf{S}}(\sigma) := \hat{\mathbf{T}}(\mathbf{H}, \mathbf{N})\mathbf{G}^{-1} \ . \tag{1.6}$$

We assume that a continuous function

$$\hat{\mathbf{F}} : \mathrm{Lin}_0\mathcal{T} \ \times \ \mathrm{Lin}_1\mathcal{T} \longrightarrow \ \mathrm{Lin}\,\mathcal{T} \tag{1.7}$$

is given which satisfies, for all $\mathbf{H} \in \mathrm{Lin}_0\mathcal{T},\ \mathbf{N} \in \mathrm{Lin}_1\mathcal{T}$,

$$\hat{\mathbf{F}}(\mathbf{AHA}^{-1}, \mathbf{ANA}^{-1}) = \mathbf{A}\hat{\mathbf{F}}(\mathbf{H}, \mathbf{N})\mathbf{A}^{-1} \quad \text{for all} \quad \mathbf{A} \in \mathrm{Unim}\mathcal{T} \ , \tag{1.8}$$

$$\hat{\mathbf{F}}(\mathbf{G}^{-1}\mathbf{H}^\top\mathbf{G}, \mathbf{G}^{-1}\mathbf{N}^\top\mathbf{G}) = \mathbf{G}^{-1}\hat{\mathbf{F}}(\mathbf{H}, \mathbf{N})^\top\mathbf{G} \quad \text{for all} \quad \mathbf{G} \in \mathcal{G} \ , \tag{1.9}$$

and

$$\mathrm{tr}\,\hat{\mathbf{F}}(\mathbf{H}, \mathbf{N}) = \mathrm{tr}\,(\mathbf{HN}) \ . \tag{1.10}$$

Proposition 2. *Let a state $\sigma_0 := (\mathbf{G}, \mathbf{H}_0, \mathbf{N}_0) \in \Sigma$ and a continuously differentiable deformation process $\mathbf{P}$ of duration $d_{\mathbf{P}}$ with $\mathbf{P}^i = \mathbf{G}$ be given. Put*

$$\bar{\mathbf{H}} := \mathbf{P}^{-1}\mathbf{P}^\bullet \tag{1.11}$$

and consider the following initial value problem: find the continuously differentiable function $\bar{\mathbf{N}} : I_{\mathbf{P}} \longrightarrow \mathrm{Lin}_0\mathcal{T}$ which satisfies

$$\bar{\mathbf{N}}^\bullet = \hat{\mathbf{F}}(\bar{\mathbf{H}}, \bar{\mathbf{N}}) - \bar{\mathbf{H}}\bar{\mathbf{N}} \quad \text{and} \quad \bar{\mathbf{N}}(0) := \mathbf{N}_0 \ . \tag{1.12}$$

(We assume that the function $\hat{\mathbf{F}}$ is such that this initial value problem always has exactly one solution $\bar{\mathbf{N}}$.) Then

$$(\mathbf{P}(t), \hat{\mathbf{H}}(t), \hat{\mathbf{N}}(t)) \in \Sigma \qquad \text{for all} \qquad t \in I_{\mathbf{P}} \ . \tag{1.13}$$

Proof: The values of $\mathbf{P}\bar{\mathbf{H}} = \mathbf{P}^{\bullet}$ all belong to $\mathrm{Sym}(\mathcal{T}, \mathcal{T}^*)$. Using Prop.9 and the differentiation rule (0.2) for determinants, both in [N4], it is easily seen that

$$\mathrm{tr}\bar{\mathbf{H}} = 0 \ . \tag{1.14}$$

i.e. that the values of $\bar{\mathbf{H}}$ belong to $\mathrm{Lin}_0\mathcal{T}$.

It follows from (1.10) and (1.12) that $(\mathrm{tr}\bar{\mathbf{N}})^{\bullet} = 0$. Since $\mathbf{N}_0 \in \mathrm{Lin}_1\mathcal{T}$ and hence $\mathrm{tr}\mathbf{N}_0 = 1$, we conclude that $\mathrm{tr}\bar{\mathbf{N}} = 1$, i.e. that the values of $\bar{\mathbf{N}}$ belong to $\mathrm{Lin}_1\mathcal{T}$.

It is an easy consequence of (1.12) and (1.11) that

$$(\mathbf{P}\bar{\mathbf{N}})^{\bullet} = \mathbf{P}\hat{\mathbf{F}}(\bar{\mathbf{H}}, \bar{\mathbf{N}}) \ . \tag{1.15}$$

We put

$$\bar{\mathbf{N}}' := \mathbf{P}^{-1}\bar{\mathbf{N}}^{\top}\mathbf{P} \ . \tag{1.16}$$

Since $\mathbf{P}^{-1}\bar{\mathbf{H}}^{\top}\mathbf{P} = \bar{\mathbf{H}}$ by (1.11), it follows from (1.16), (1.15) and (1.9) that

$$(\mathbf{P}\bar{\mathbf{N}}')^{\bullet} = (\bar{\mathbf{N}}^{\top}\mathbf{P})^{\bullet} = ((\mathbf{P}\bar{\mathbf{N}})^{\top})^{\bullet} = \hat{\mathbf{F}}(\bar{\mathbf{H}}, \bar{\mathbf{N}})^{\top}\mathbf{P} =$$

$$\mathbf{P}\hat{\mathbf{F}}(\mathbf{P}^{-1}\bar{\mathbf{H}}^{\top}\mathbf{P}, \mathbf{P}^{-1}\bar{\mathbf{N}}^{\top}\mathbf{P}) = \mathbf{P}\hat{\mathbf{F}}(\bar{\mathbf{H}}, \bar{\mathbf{N}}') \ . \tag{1.17}$$

Now, since $(\mathbf{P}\bar{\mathbf{N}})(0) = \mathbf{G}\mathbf{N}_0 \in \mathrm{Sym}(\mathcal{T}, \mathcal{T}^*)$, it follows from (1.16) that $\bar{\mathbf{N}}'(0) = \bar{\mathbf{N}}(0) = \mathbf{N}_0$. Therefore, by (1.17), both $\bar{\mathbf{N}}$ and $\bar{\mathbf{N}}'$ are solutions of the same initial value problem and hence equal. In view of (1.16), this means that the values of $\mathbf{P}\bar{\mathbf{N}}$ all belong to $\mathrm{Sym}(\mathcal{T}, \mathcal{T}^*)$. ∎

In view of Prop.2 it is meaningful to specify that the evolution function $\hat{\rho}$ have the value

$$\hat{\rho}(\sigma_0, \mathbf{P}) := ((\mathbf{P}(d_{\mathbf{P}}), \bar{\mathbf{H}}(d_{\mathbf{P}}), \bar{\mathbf{N}}(d_{\mathbf{P}})) \ , \tag{1.18}$$

when $\mathbf{P}$ is of finite duration $d_{\mathbf{P}}$.

It must be noted that the class of continuously differentiable deformation processes is not stable under continuation and hence does not satisfy the axiom (P2) of Sect.7 of [3]. Hence we must enlarge the class of deformation processes by admitting those that are continuously differentiable except at a finite set of times and whose derivatives have jump-discontinuities at these times. This class satisfies the axioms (P1), (P2), and (E1) of Sect.7 in [3]. Moreover, the formula (7.9) of [3] shows that there is exactly one way of extending the evolution function given by (1.18) to processes in this enlarged class in such a way that the axiom (E2) is also satisfied.

Let $\mathbf{A} \in \mathrm{Unim}\,\mathcal{T}$ be given and define $\iota : \mathrm{Unim}\,\mathcal{T} \longrightarrow \mathrm{Perm}\,\Sigma$ by

$$\iota_{\mathbf{A}}(\mathbf{G},\mathbf{H},\mathbf{N}) := (\mathbf{A}^{\top -1}\mathbf{G}\mathbf{A}^{-1}, \mathbf{A}\mathbf{H}\mathbf{A}^{-1}, \mathbf{A}\mathbf{N}\mathbf{A}^{-1}) \quad \text{for all} \quad (\mathbf{G},\mathbf{H},\mathbf{N}) \in \Sigma\ . \tag{1.19}$$

Since $\mathbf{A} \in \mathrm{Unim}\,\mathcal{T}$ was arbitrary, it is then easy to show, using (1.4), (1.6), and (1.8), that ι is an action of $\mathrm{Unim}\,\mathcal{T}$ on Σ having the properties (8.8), (8.9), and (8.10) of [3]. Hence, the symmetry group is the unimodular group and the material element is semi-liquid in the sense of Def.1 in Sect.8 of [3].

Remark 1: Suppose that the function (1.3) is assumed to satisfy (1.4), but (1.5) only for *some* $\mathbf{G} \in \mathcal{G}$ instead of *for all* $\mathbf{G} \in \mathcal{G}$. It is then easy to prove, using (1.4) and Prop.9 of [4], that (1.5) must in fact be valid for all $\mathbf{G} \in \mathcal{G}$. Of course, an analagous statement applies to the function (1.7). ∎

Remark 2: The assumption (1.4) is actually a consequence of (1.5) and hence redundant. To see this, let $\mathbf{G}_1, \mathbf{G}_2 \in \mathcal{G}$ be given and put $\mathbf{A} := \mathbf{G}_2^{-1}\mathbf{G}_1$. Using (1.5) with $\mathbf{G}$ replaced by $\mathbf{G}_1$ and then again with $\mathbf{G}$ replaced by $\mathbf{G}_2$ and $\mathbf{H}$ by $\mathbf{G}_1^{-1}\mathbf{H}\mathbf{G}_2$, it is easily seen that (1.4) is valid for the given $\mathbf{A}$.

Now, in view of Prop.9 in Sect.4 of [4], the set of all $\mathbf{A}$ of the form $\mathbf{A} := \mathbf{G}_2^{-1}\mathbf{G}_1$ with $\mathbf{G}_1, \mathbf{G}_2 \in \mathcal{G}$ consists of all lineons in $\mathrm{Unim}\mathcal{T}$ that are diagonable and have positive spectral values. It turns out that this set, together with $-\mathbf{1}_{\mathcal{T}}$, generates the entire unimodular group $\mathrm{Unim}\mathcal{T}$. (William Lawvere of SUNY Buffalo informed me that his collegue Stephen H. Schanual has a proof of this fact). Therefore (1.4) follows from (1.5). Similarly, (1.8) is is a consequence of (1.9). ∎

2. The special model.

We now consider the case when the functions $\hat{\mathbf{T}}$ and $\hat{\mathbf{F}}$ have the following special form:

$$\hat{\mathbf{T}}(\mathbf{H},\mathbf{N}) := (\lambda_1 + \frac{\lambda_2}{2}\mathrm{tr}(\mathbf{H}\mathbf{N}))\mathbf{N} + \frac{\lambda_3}{2}\mathbf{H} + \frac{\lambda_4}{2}(\mathbf{H}\mathbf{N} + \mathbf{N}\mathbf{H}) \tag{2.1}$$

and

$$\hat{\mathbf{F}}(\mathbf{H},\mathbf{N}) := (\mu+1)(\mathbf{H}\mathbf{N} + \mathbf{N}\mathbf{H}) - (2\mu+1)\mathrm{tr}(\mathbf{H}\mathbf{N})\mathbf{N}\ , \tag{2.2}$$

valid for all $(\mathbf{H},\mathbf{N}) \in \mathrm{Lin}_0\mathcal{T} \times \mathrm{Lin}_1\mathcal{T}$. The numbers $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \mu\ \in \mathbb{R}$ are material constants. It is easy to verify that (1.4), (1.5), (1.8), (1.9), and (1.10) are all satisfied when (2.1) and (2.2) hold.

As in the previous section, we now assume that an initial state $\sigma_0 := (\mathbf{G},\mathbf{H}_0,\mathbf{N}_0) \in \Sigma$ and a continuously differentiable deformation process $\mathbf{P}$ of duration $d_{\mathbf{P}}$ with $\mathbf{P}^i = \mathbf{G}$ are given and we use the notation (1.11). The initial value problem of Prop.1 then becomes:

Find the continuously differentiable function $\bar{\mathbf{N}} : I_{\mathbf{P}} \longrightarrow \mathrm{Lin}_0\mathcal{T}$ which satisfies

$$\bar{\mathbf{N}}^{\bullet} = \mu\bar{\mathbf{H}}\bar{\mathbf{N}} + (\mu+1)\bar{\mathbf{N}}\bar{\mathbf{H}} - (2\mu+1)\mathrm{tr}(\bar{\mathbf{H}}\bar{\mathbf{N}})\bar{\mathbf{N}} \quad \text{and} \quad \bar{\mathbf{N}}(0) := \mathbf{N}_0\ . \tag{2.3}$$

The following result shows that one can reduce the initial-value problem (2.3) to a simpler one if the initial value of $\mathbf{N}_0$ is idempotent, i.e. if $\mathbf{N}_0^2 = \mathbf{N}_0$.

Proposition 3. *Assume that $\mathbf{N}_0$ is idempotent. In that case, the values of the solution $\bar{\mathbf{N}}$ of the initial value problem (2.3) are all idempotent. Moreover, there is a $\mathbf{n}_0 \in \mathcal{T}$ such that the solution of the initial-value problem: Find $\mathbf{n} : I_{\mathbf{P}} \longrightarrow \mathcal{T}$, of class C^1, such that*

$$\mathbf{n}^\bullet = \mu\bar{\mathbf{H}}\mathbf{n} - (\mu + \frac{1}{2})((\mathbf{P}\mathbf{n})(\bar{\mathbf{H}}\mathbf{n}))\mathbf{n} \quad \text{and} \quad \mathbf{n}(0) = \mathbf{n}_0 \tag{2.4}$$

is related to the solution $\bar{\mathbf{N}}$ of the initial-value problem (2.3) by

$$\bar{\mathbf{N}} = \mathbf{n} \otimes \mathbf{P}\mathbf{n} \ . \tag{2.5}$$

Proof: First, using Prop.1, we choose $\mathbf{n}_0 \in \mathcal{T}$ such that

$$\mathbf{N}_0 = \mathbf{n}_0 \otimes \mathbf{G}\mathbf{n}_0 \quad \text{and} \quad (\mathbf{G}\mathbf{n}_0)\mathbf{n}_0 = 1 \tag{2.6}$$

and we consider the solution $\mathbf{n} : I_{\mathbf{P}} \longrightarrow \mathcal{T}$ of the initial value problem (2.4). We now *define* $\bar{\mathbf{N}}$ by (2.5). We then have

$$(\mathbf{P}\mathbf{n})\bar{\mathbf{H}}\mathbf{n} = \mathrm{tr}(\bar{\mathbf{H}}\mathbf{n} \otimes \mathbf{P}\mathbf{n}) = \ \mathrm{tr}(\bar{\mathbf{H}}(\mathbf{n} \otimes \mathbf{P}\mathbf{n}) = \mathrm{tr}(\bar{\mathbf{H}}\bar{\mathbf{N}}) = \mathrm{tr}(\bar{\mathbf{N}}\bar{\mathbf{H}}).$$

Hence (2.4) gives

$$\mathbf{n}^\bullet = \mu\mathbf{H}\mathbf{n} - (\mu + \frac{1}{2})(\mathrm{tr}(\bar{\mathbf{N}}\bar{\mathbf{H}}))\mathbf{n}. \tag{2.7}$$

Using (2.5), it follows that

$$\mathbf{n}^\bullet \otimes \mathbf{P}\mathbf{n} = \mu\bar{\mathbf{H}}\bar{\mathbf{N}} - (\mu + \frac{1}{2})(\mathrm{tr}(\bar{\mathbf{N}}\bar{\mathbf{H}}))\bar{\mathbf{N}}. \tag{2.8}$$

It follows from (1.11) that $\mathbf{P}^\bullet = \mathbf{P}^{\bullet\top} = (\mathbf{P}\bar{\mathbf{H}})^\top = \bar{\mathbf{H}}^\top\mathbf{P}$ and hence,by (2.5),

$$\mathbf{n} \otimes \mathbf{P}^\bullet\mathbf{n} = \mathbf{n} \otimes (\bar{\mathbf{H}}^\top\mathbf{P})\mathbf{n} = (\mathbf{n} \otimes \mathbf{P}\mathbf{n})\bar{\mathbf{H}} = \bar{\mathbf{N}}\bar{\mathbf{H}}. \tag{2.9}$$

Using (2.7), we see that

$$\mathbf{n} \otimes \mathbf{P}\mathbf{n}^\bullet = \mu(\mathbf{n} \otimes \mathbf{P}\bar{\mathbf{H}}\mathbf{n}) - (\mu + \frac{1}{2})(\mathrm{tr}(\bar{\mathbf{N}}\bar{\mathbf{H}})(\mathbf{n} \otimes \mathbf{P}\mathbf{n})$$

Therefore, since $\mathbf{P}\bar{\mathbf{H}} = \mathbf{P}^\bullet$ by (1.11), it follows from (2.9) and (2.5) that

$$\mathbf{n} \otimes \mathbf{P}\mathbf{n}^\bullet = \mu\bar{\mathbf{N}}\bar{\mathbf{H}} - (\mu + \frac{1}{2})(\mathrm{tr}(\bar{\mathbf{N}}\bar{\mathbf{H}})\bar{\mathbf{N}}. \tag{2.10}$$

Differentiating (2.5) using the product rule gives

$$\bar{\mathbf{N}}^\bullet = \mathbf{n}^\bullet \otimes \mathbf{P}\mathbf{n} + \mathbf{n} \otimes \mathbf{P}^\bullet\mathbf{n} + \mathbf{n} \otimes \mathbf{P}\mathbf{n}^\bullet. \tag{2.11}$$

Substituting (2.8), (2.9), and (2.10) for the three terms on the right side of (2.11), we easily sees that $\bar{\mathbf{N}}$ does indeed satisfy the differential equation (2.3). ∎

Remark 3: Prop.3 shows that one could reduce the size of the state space Σ by adding the condition $\mathbf{N}^2 = \mathbf{N}$ in (1.1) and still use the differental equation (1.12) to define the evolution function $\hat{\sigma}$ by (1.18). In this case, the special model becomes a frame-free description of a theory first introduced in 1960 by J.L. Ericksen with the name "Theory of incompressible anisotropic fluids". (See Sects. 127, 128, and 129 in [NLFT] and the literature cited there.) In fact, this is a theory of semi-liquid material elements with no isotropic states in the sense of the definition given in Sect.8 of [3], so that the term "anisotropic fluid" is not entirely inappropriate, even though it is only a semi-fluid in the present terminology. We note that (2.1), with $\mathbf{N}^2 = \mathbf{N}$, is a frame-free version of the constitutive equation (127.34) in [NLFT]. ∎

We now consider the case when the given deformation process $\mathbf{P}$ in Prop.2 is obtained from a motion

$$\mathbf{M} : I_{\mathbf{P}} :\longrightarrow \mathrm{Lis}\,(\mathcal{T}, \mathcal{V})$$

in a frame-space $\mathcal{F}$ with translation $\mathcal{V}$ space as described in Sect.4 of [N3]. We have

$$\mathbf{P} := \mathbf{M}^\top \mathbf{M} : \ I_{\mathbf{P}} \longrightarrow \mathcal{G} \ . \tag{2.12}$$

We assume that $\mathbf{M}$ is differentiable. Its derivative is given by

$$\mathbf{M}^\bullet = \mathbf{L}\mathbf{M}, \tag{2.13}$$

where the value of $\mathbf{L} : [0, d_{\mathbf{P}}] \longrightarrow \mathrm{Lin}\,(\mathcal{V})$ at time t gives the velocity gradient of the body at the location of the material element at that time. Recall that the values of the symmetric and skew parts of $\mathbf{L}$, i. e.

$$\mathbf{D} := \frac{1}{2}(\mathbf{L} + \mathbf{L}^\top) \ , \quad \mathbf{W} := \frac{1}{2}(\mathbf{L} - \mathbf{L}^\top) \ , \quad \text{with} \ \ \mathbf{L} = \mathbf{D} + \mathbf{W} \tag{2.14}$$

give the corresponding stretching and spin, respectively. By (4.7) in [N3] and using (2.12) and the definition (1.11) above, we find

$$\bar{\mathbf{H}} = \mathbf{P}^{-1}\mathbf{P}^\bullet = 2\mathbf{M}^{-1}\mathbf{D}\mathbf{M}. \tag{2.15}$$

Let $\bar{\mathbf{N}}$ be the solution of the initial value problem (1.12) of Prop.2. A spatial counterpart $\bar{\mathbf{N}}$ is given by

$$\mathbf{V} := \mathbf{M}\bar{\mathbf{N}}\mathbf{M}^{-1} : I_{\mathbf{P}} \longrightarrow \mathrm{Sym}\mathcal{V} \cap \mathrm{Lin}_1\mathcal{V} \ . \tag{2.16}$$

Since, by Prop.2, $\mathbf{P}\bar{\mathbf{N}}$ has values in $\mathrm{Sym}(\mathcal{T}, \mathcal{T}^*)$ it easily follows from (2.12) that the values of $\mathbf{V}$ are indeed symmetric. These values have trace 1 because the

values of $\bar{\mathbf{N}}$ have trace 1. An easy calculation, using the product rule and (2.13) shows that

$$\mathbf{V}^\bullet = \mathbf{LV} - \mathbf{VL} + \mathbf{M}\bar{\mathbf{N}}^\bullet\mathbf{M}^{-1}. \tag{2.17}$$

Assuming that the differential equation (2.3) for the special model is satisfied, we substitute (2.3) for $\bar{\mathbf{N}}^\bullet$ in (2.17), observe (2.15) and (2.16), and obtain the following differential equation for $\mathbf{V}$

$$\mathbf{V}^\bullet = \mathbf{LV} - \mathbf{VL} + 2\mu\mathbf{DV} + 2(\mu+1)\mathbf{VD} - 2(2\mu+1)\mathrm{tr}(\mathbf{DV})\mathbf{V}\ . \tag{2.18}$$

Since $\mathbf{L} = \mathbf{D} + \mathbf{W}$ by (2.14), this differential equation can also be written in the form

$$\mathbf{V}^\bullet + \mathbf{VW} - \mathbf{WV} = 2(\mu+1)(\mathbf{DV} + \mathbf{VD}) - 2\mathrm{tr}(\mathbf{DV})\mathbf{V}\ . \tag{2.19}$$

Consider now the case when the values of $\bar{\mathbf{N}}$ are idempotent, as described in Prop.3. In view of (2.16), the values of $\mathbf{V}$ are then also idempotent and we have $\mathbf{V} = \mathbf{d}\otimes\mathbf{d}$, where $\mathbf{d} : d_{\mathbf{P}} :\longrightarrow \mathcal{V}$, in view of (2.12), is related to the $\mathbf{n}$ of Prop.3 by

$$\mathbf{d} := \mathbf{Mn}. \tag{2.20}$$

The values of $\mathbf{d}$ are unit vectors because $\mathrm{tr}\mathbf{V} = \mathbf{d}\bullet\mathbf{d} = 1$. Using the product rule and (2.13), we obtain from (2.20) that

$$\mathbf{d}^\bullet = \mathbf{Mn}^\bullet + \mathbf{M}^\bullet\mathbf{n} = \mathbf{Mn}^\bullet + \mathbf{LMn} = \mathbf{Mn}^\bullet + \mathbf{Ld}\ . \tag{2.21}$$

substituting (2.4) for $\mathbf{n}^\bullet$ in (2.21) and using (2.12), $(2.15)_2$, and (2.14), a short calculation gives the following differential equation for $\mathbf{d}$:

$$\mathbf{d}^\bullet - \mathbf{Wd} = (2\mu+1)(\mathbf{Dd} - (\mathbf{d}\bullet\mathbf{Dd})\mathbf{d}). \tag{2.22}$$

Remark 4: The equation (2.22) here is the same as (127.36) in [NLFT], with λ replaced by $(2\mu+1)$ and $\overset{\circ}{\mathbf{d}}$ by $(127.31)_1$ in [NLFT]. It is the coordinate-free version of the equations first proposed be J.L. Ericksen as mentioned in Remark 3. ∎

3. Monotonous Processes.

We first consider the general model of Sect.1 and assume that a configuration $\mathbf{G} \in \mathcal{G}$ and a monotonous process $\mathbf{P}$ of exponent $\mathbf{E} \in \mathrm{Lin}_0\mathcal{T}$ with $\mathbf{P}^i = \mathbf{G}$ are given (see Sect.1 of [4]). By the Representation Theorem of Monotonous Processes in Sect.1 of [N4] we then have

$$\mathbf{P} = \mathbf{L}^\top\mathbf{GL} \quad : \quad I_{\mathbf{P}} \longrightarrow \mathcal{G} \tag{3.1}$$

with

$$\mathbf{L}(t) := \exp(t\mathbf{E}) \qquad \text{for all} \qquad t \in I_{\mathbf{P}} \tag{3.2}$$

and hence

$$\mathbf{L}^\bullet = \mathbf{LE} = \mathbf{EL}\ . \tag{3.3}$$

As in the previous section, we also assume that $\mathbf{N}_0 \in \mathrm{Lin}_1\mathcal{T}$ is given and we observe the definition (1.11) of $\bar{\mathbf{H}}$.

Proposition 4. *Let $\bar{\mathbf{N}}$ be the solution of the initial value problem (1.12).Then*

$$\bar{\mathbf{K}} := \mathbf{L}\bar{\mathbf{N}}\mathbf{L}^{-1} \tag{3.4}$$

is the solution to the initial value problem

$$\bar{\mathbf{K}}^{\bullet} = \hat{\mathbf{F}}(\mathbf{G}^{-1}\mathbf{E}^{\top}\mathbf{G} + \mathbf{E}, \bar{\mathbf{K}}) - (\mathbf{G}^{-1}\mathbf{E}^{\top}\mathbf{G}\bar{\mathbf{K}} + \bar{\mathbf{K}}\mathbf{E}), \quad \bar{\mathbf{K}}(0) = \mathbf{N}_0 \ . \tag{3.5}$$

Proof : It follows from (3.1), the product rule, and (3.3) that

$$\mathbf{P}^{\bullet} = (\mathbf{L}^{\bullet})^{\top}\mathbf{G}\mathbf{L} + \mathbf{L}^{\top}\mathbf{G}\mathbf{L}^{\bullet} = \mathbf{L}^{\top}(\mathbf{E}\mathbf{G} + \mathbf{G}\mathbf{E})\mathbf{L} \ ,$$

and hence, by (3.1) and (1.11), that

$$\bar{\mathbf{H}} = (\mathbf{L}^{\top}\mathbf{G}\mathbf{L})^{-1}\mathbf{P}^{\bullet} = \mathbf{L}^{-1}(\mathbf{G}^{-1}\mathbf{E}^{\top}\mathbf{G} + \mathbf{E})\mathbf{L} \ . \tag{3.6}$$

Using the condition (1.8) on $\hat{\mathbf{F}}$ with $\mathbf{A} := \mathbf{L}(t)$ for every $t \in I_{\mathbf{P}}$, it follows from (3.4) and (3.6) that

$$\hat{\mathbf{F}}(\bar{\mathbf{H}}, \bar{\mathbf{N}}) = \mathbf{L}^{-1}\hat{\mathbf{F}}(\mathbf{G}^{-1}\mathbf{E}^{\top}\mathbf{G} + \mathbf{E}, \bar{\mathbf{K}})\mathbf{L} \ . \tag{3.7}$$

By (3.4) we have $\bar{\mathbf{K}}\mathbf{L} = \mathbf{L}\bar{\mathbf{N}}$. Differentiating this equation using the product rule and (3.3), we find that

$$\bar{\mathbf{K}}^{\bullet}\mathbf{L} + \bar{\mathbf{K}}\mathbf{E}\mathbf{L} = \mathbf{E}\mathbf{L}\bar{\mathbf{N}} + \mathbf{L}\bar{\mathbf{N}}^{\bullet} = \mathbf{E}\bar{\mathbf{K}}\mathbf{L} + \mathbf{L}\bar{\mathbf{N}}^{\bullet}. \tag{3.8}$$

Using (1.12), (3.6), (3.7) and (3.4), a short calculation shows that the desired result (3.5) is valid. ∎

It follows from the conclusion (1.13) of Prop.1 and from (1.1) that the values of $\mathbf{P}\bar{\mathbf{N}}$ belong to $\mathrm{Sym}(\mathcal{T}, \mathcal{T}^*)$, i.e. that $\mathbf{P}\bar{\mathbf{N}} = (\mathbf{P}\bar{\mathbf{N}})^{\top}$. Using (3.1) and (3.4), it follows that

$$\mathbf{G}\bar{\mathbf{K}} = \bar{\mathbf{K}}^{\top}\mathbf{G}. \tag{3.9}$$

From now on we use $\mathbf{G}$ to endow $\mathcal{T}$ with the structure of an inner-product space, so that $\mathcal{T}^*$ becomes identified with $\mathcal{T}$ via the isomorphism $\mathbf{G}$. Then $\mathrm{Sym}\,(\mathcal{T}, \mathcal{T}^*)$ becomes identified with $\mathrm{Sym}\,\mathcal{T} := \mathrm{Sym}\,(\mathcal{T}, \mathcal{T})$, which is a subspace of $\mathrm{Lin}\,\mathcal{T}$. The formulas (3.9), (3.6) and (3.5) then remain valid when $\mathbf{G}$ is replaced by the identity $\mathbf{1}_{\mathcal{T}}$. Therefore $\bar{\mathbf{K}}$ has values in $\mathrm{Sym}\,\mathcal{T}$ and (3.6) reduces to

$$\bar{\mathbf{H}} = \mathbf{L}^{-1}(\mathbf{E}^{\top} + \mathbf{E})\mathbf{L}. \tag{3.10}$$

Using (1.9) for the particular $\mathbf{G}$ given in this section, we see that $\hat{\mathbf{F}}(\mathbf{H}, \mathbf{N}) \in \mathrm{Sym}\,\mathcal{T}$ when $\mathbf{H}, \mathbf{N} \in \mathrm{Sym}\,\mathcal{T}$. Putting

$$\mathrm{Sym}_c\mathcal{T} := \mathrm{Lin}_c\mathcal{T} \ \cap \ \mathrm{Sym}\,\mathcal{T} \quad \text{when} \ \ c := 0 \text{ or } 1, \tag{3.11}$$

we see that we can replace $\hat{\mathbf{F}}$ in (3.5) by an adjusted mapping

$$\hat{\mathbf{F}}_{\mathbf{G}} : \ \mathrm{Sym}_0\mathcal{T} \ \times \ \mathrm{Sym}_1\mathcal{T} \ \longrightarrow \ \mathrm{Sym}_0\mathcal{T} \tag{3.12}$$

and hence reduce (3.5) to

$$\bar{\mathbf{K}}^{\bullet} = \hat{\mathbf{F}}_{\mathbf{G}}(\mathbf{E}^{\top} + \mathbf{E}, \bar{\mathbf{K}}) - (\mathbf{E}^{\top}\bar{\mathbf{K}} + \bar{\mathbf{K}}\mathbf{E}), \quad \bar{\mathbf{K}}(0) = \mathbf{N}_0. \tag{3.13}$$

It follows from (1.8) that $\hat{\mathbf{F}}_{\mathbf{G}}$ satisfies, for all $\mathbf{H} \in \mathrm{Sym}_0\mathcal{T}$, $\mathbf{N} \in \mathrm{Sym}_1\mathcal{T}$,

$$\hat{\mathbf{F}}_{\mathbf{G}}(\mathbf{Q}\mathbf{H}\mathbf{Q}^{\top}, \mathbf{Q}\mathbf{N}\mathbf{Q}^{\top}) = \mathbf{Q}\hat{\mathbf{F}}_{\mathbf{G}}(\mathbf{H}, \mathbf{N})\mathbf{Q}^{\top} \quad \text{for all} \quad \mathbf{Q} \in \mathrm{Orth}\mathcal{T} \ . \tag{3.14}$$

By (3.2) we have $\mathbf{L}(0) = \mathbf{1}_{\mathcal{T}}$ and hence, by (3.10),

$$\mathbf{H}_0 := \bar{\mathbf{H}}(0) = \mathbf{E}^{\top} + \mathbf{E} \ . \tag{3.15}$$

Proposition 5. *Put*

$$\sigma_0 := (\mathbf{G}, \mathbf{H}_0, \mathbf{N}_0). \tag{3.16}$$

Then

$$\iota_{\mathbf{L}(t)}\rho(\sigma_0, \mathbf{P}_{[0,t]}) = (\mathbf{G}, \mathbf{E}^{\top} + \mathbf{E}, \bar{\mathbf{K}}(t)) \qquad \text{for all} \quad t \in I\!P, \tag{3.17}$$

and σ_0 is a monotonous state if and only if $\bar{\mathbf{K}}(t)$ has the constant value $\mathbf{N}_0$, i.e., when

$$\hat{\mathbf{F}}_{\mathbf{G}}(\mathbf{E}^{\top} + \mathbf{E}, \mathbf{N}_0) = (\mathbf{E}^{\top}\mathbf{N}_0 + \mathbf{N}_0\mathbf{E}) \ . \tag{3.18}$$

Proof: It follows from (1.18), (3.1), (3.10), and (3.4) that

$$\rho(\sigma_0, \mathbf{P}_{[0,t]}) = (\mathbf{P}(t), \bar{\mathbf{H}}(t), \bar{\mathbf{N}}(t)) =$$

$$= (\mathbf{L}(t)^{\top}\mathbf{G}\mathbf{L}(t), \ \mathbf{L}(t)^{-1}(\mathbf{E}^{\top} + \mathbf{E})\mathbf{L}(t), \ \mathbf{L}(t)^{-1}\bar{\mathbf{K}}(t)\mathbf{L}(t)) \ . \tag{3.19}$$

Using (1.19) with $\mathbf{A} := \mathbf{L}(t)$ for every $t \in \mathbb{P}$, we conclude that (3.17) is valid. The condition for σ_0 to be a monotonous state is in accord with Def.2 of Sect.2 in [N4]. ∎

We now consider the case of the special model described in Sect.2, i.e. the case when $\hat{\mathbf{T}}$ and $\hat{\mathbf{F}}$ have the forms (2.1) and (2.2). Using (2.2) and recalling that $\bar{\mathbf{K}}$ has values in $\mathrm{Sym}\,\mathcal{T}$, we easily see that the initial value problem (3.13) reduces to

$$\bar{\mathbf{K}}^{\bullet} = \mathbf{C}\bar{\mathbf{K}} + \bar{\mathbf{K}}\mathbf{C}^{\top} - 2\mathrm{tr}(\mathbf{C}\bar{\mathbf{K}})\bar{\mathbf{K}} \ , \quad \bar{\mathbf{K}}(0) = \mathbf{N}_0 \ , \tag{3.20}$$

where

$$\mathbf{C} := (\mu + 1)\mathbf{E} + \mu\mathbf{E}^{\top} \ . \tag{3.21}$$

It turns out that this initial value problem can be solved explicitly:

Proposition 6. *The solution of the initial value problem (3.20) is*

$$\bar{\mathbf{K}} = \frac{\mathbf{J}\mathbf{N}_0\mathbf{J}^\top}{\mathrm{tr}(\mathbf{J}\mathbf{N}_0\mathbf{J}^\top)}, \tag{3.22}$$

where $\mathbf{J} : I\!P \longrightarrow \mathrm{Lin}\,\mathcal{T}$ *is defined by*

$$\mathbf{J}(t) := \exp(t\mathbf{C}) \qquad \text{for all} \quad t \in I\!P. \tag{3.23}$$

Proof: It is clear from (3.23) that

$$\mathbf{J}^\bullet = \mathbf{C}\mathbf{J} = \mathbf{J}\mathbf{C}. \tag{3.24}$$

We now *define* $\bar{\mathbf{K}}$ by (3.22) and put

$$\tau := \mathrm{tr}(\mathbf{J}\mathbf{N}_0\mathbf{J}^\top), \tag{3.25}$$

so that

$$\tau\bar{\mathbf{K}} = (\mathbf{J}\mathbf{N}_0\mathbf{J}^\top)\ . \tag{3.26}$$

By the product rule, (3.24), and (3.26) we have

$$(\mathbf{J}\mathbf{N}_0\mathbf{J}^\top)^\bullet = \mathbf{J}^\bullet\mathbf{N}_0\mathbf{J}^\top + \mathbf{J}\mathbf{N}_0\mathbf{J}^{\bullet\top} = \tau(\mathbf{C}\bar{\mathbf{K}} + \bar{\mathbf{K}}\mathbf{C}^\top)\ , \tag{3.27}$$

and hence, by (3.25),

$$\tau^\bullet = \tau\ \mathrm{tr}(\mathbf{C}\bar{\mathbf{K}} + \bar{\mathbf{K}}\mathbf{C}^\top) = 2\tau\ \mathrm{tr}(\mathbf{C}\bar{\mathbf{K}})\ . \tag{3.28}$$

Applying the product rule to (3.26) and observing (3.27), we see that

$$\tau^\bullet\bar{\mathbf{K}} + \tau\bar{\mathbf{K}}^\bullet = \tau(\mathbf{C}\bar{\mathbf{K}} + \bar{\mathbf{K}}\mathbf{C}^\top),$$

which, by (3.28), shows that (3.20) is valid. ∎

4. Monotonous states.

We assume that a configuration $\mathbf{G} \in \mathcal{G}$ is given and, as after (3.9), we use $\mathbf{G}$ to endow $\mathcal{T}$ with the structure of an inner product space. Also, we let an exponent $\mathbf{E} \in \mathrm{Lin}_0\mathcal{T}$ be given.

Proposition 7. *Let* $\mathbf{H} \in \mathrm{Sym}_0\mathcal{T},\ \mathbf{N} \in \mathrm{Sym}_1\mathcal{T}$ *be given. Then*

$$\sigma := (\mathbf{G}, \mathbf{H}, \mathbf{N}). \tag{4.1}$$

is a monotonous state of exponent $\mathbf{E}$ *(in the sense of Def.2 of [N4]) if and only if*

$$\mathbf{H} = \mathbf{E}^\top + \mathbf{E} \tag{4.2}$$

and

$$\hat{\mathbf{F}}(\mathbf{E}^\top + \mathbf{E}, \mathbf{N}) = (\mathbf{E}^\top \mathbf{N} + \mathbf{N}\mathbf{E}) \ . \tag{4.3}$$

For the special model, (4.3) reduces to

$$\mathbf{C}\mathbf{N} + \mathbf{N}\mathbf{C}^\top - 2\mathrm{tr}(\mathbf{C}\mathbf{N})\mathbf{N} = 0, \tag{4.4}$$

where

$$\mathbf{C} := (\mu + 1)\mathbf{E} + \mu\mathbf{E}^\top \tag{4.5}$$

Proof : Apply Prop.5 of the previous section, with $\sigma_0 := (\mathbf{G}, \mathbf{H}_0, \mathbf{N}_0)$ replaced by $\sigma := (\mathbf{G}, \mathbf{H}, \mathbf{N})$, to the monotonous process $\mathbf{P}$ of infinite duration, of exponent $\mathbf{E}$, and initial configuration $\mathbf{G}$. Comparing (2.2) of Def.2 [4] with (3.17), we see that σ is a monotonous state if and only (4.2) holds and $\bar{\mathbf{K}}$ has the constant value $\mathbf{N}$. By (3.13), this is the case if and only if (4.3) is valid. For the special model, it follows from (3.20) and (3.21) that (4.3) is equivalent with (4.4) and (4.5). ∎

From now on, we consider only the special model. In that case, finding the monotonous states amounts to solving the eqation (4.4) for $\mathbf{N}$ with $\mathbf{C}$ defined by (4.5).

Proposition 8. *Let* $\mathbf{b}, \mathbf{c}$ *be spectral vectors of* $\mathbf{C}$ *such that* $\mathbf{b} \bullet \mathbf{c} \neq 0$. *Then* $(\mathbf{G}, \mathbf{E}^\top + \mathbf{E}, \mathbf{N})$ *is a monotonous state when*

$$\mathbf{N} := \frac{\mathbf{b} \otimes \mathbf{c} + \mathbf{c} \otimes \mathbf{b}}{2\mathbf{b} \bullet \mathbf{c}} \ . \tag{4.6}$$

Proof : Let β and γ be the spectral values of $\mathbf{C}$ corresponding to $\mathbf{b}$ and $\mathbf{c}$, respectively, so that

$$\mathbf{C}\mathbf{b} = \beta\mathbf{b}, \qquad \mathbf{C}\mathbf{c} = \gamma\mathbf{c}. \tag{4.7}$$

By (4.6) we then have

$$\mathbf{C}\mathbf{N} = \frac{\beta\mathbf{b} \otimes \mathbf{c} + \gamma\mathbf{c} \otimes \mathbf{b}}{2\mathbf{b} \bullet \mathbf{c}} \ . \tag{4.8}$$

Since $\mathrm{tr}(\mathbf{b} \otimes \mathbf{c}) = \mathrm{tr}(\mathbf{c} \otimes \mathbf{b}) = \mathbf{b} \bullet \mathbf{c}$, we see that from (4.8) $\mathrm{tr}(\mathbf{C}\mathbf{N}) = \beta + \gamma$. Since $(\mathbf{C}\mathbf{N})^\top = \mathbf{N}\mathbf{C}^\top$, a short calculation using (4.8) shows that (4.4) is indeed valid. ∎

Remark 5: It seems likely that for *every* monotonous state $(\mathbf{G}, \mathbf{E}^\top + \mathbf{E}, \mathbf{N})$, $\mathbf{N}$ must be of the form (4.6). ∎

In the case when $\mathbf{b} = \mathbf{c}$, Prop.8 reduces to:

Corollary. *Let* $\mathbf{b}$ *be a spectral unit vector of* $\mathbf{C}$, *so that* $\mathbf{b} \bullet \mathbf{b} = 1$. *Then* $(\mathbf{G}, \mathbf{E}^\top + \mathbf{E}, \mathbf{b} \otimes \mathbf{b})$ *is a monotonous state.*

We now assume that $\mathbf{C}$ is diagonable, which means that there is a basis of $\mathcal{T}$ all of whose terms are spectral vectors. (See Sect.82 of [FDS].)

Proposition 9. *Denote the largest spectal value of* $\mathbf{C}$ *by* β *and assume that the multiplicity of* β *is 1. Let* $\mathbf{b}$ *be a spectral unit vector for* β*. Then* $(\mathbf{G}, \mathbf{E}^\top + \mathbf{E}, \mathbf{b} \otimes \mathbf{b})$ *is the only stable monotonous state with exponent* $\mathbf{E}$ *and hence uni-monotonous (in the sense of Def.3 and Def.4 of [4]).*

Proof : We use the notations

$$3^] := \{1, 2, 3\} \quad \text{and} \quad 3^] \bigcirc 3^] := \{(i, k) \in 3^] \times 3^] \mid i \geq k\} \ . \tag{4.9}$$

We choose a basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ with $\mathbf{b}_1 := \mathbf{b}$ such as

$$\mathbf{C}\mathbf{b}_i = \gamma_i \mathbf{b}_i \quad \text{for all} \quad i \in 3^] \quad \text{with} \quad \gamma_1 := \beta \quad \text{and } \beta > \gamma_2, \gamma_3 \ , \tag{4.10}$$

The family $(\frac{1}{2}(\mathbf{b}_i \otimes \mathbf{b}_k + \mathbf{b}_k \otimes \mathbf{b}_i) \mid (i, k) \in 3^] \bigcirc 3^])$ is a basis of $\mathrm{Sym}\mathcal{T}$. Therefore, given $\mathbf{N}_0 \in \mathrm{Sym}_0\mathcal{T}$, we can determine a family $(N_{i,k} \mid (i, k) \in 3^] \bigcirc 3^])$ such that

$$\mathbf{N} = \sum_{(i,k) \in 3^] \bigcirc 3^]} \frac{1}{2} N_{i,k} (\mathbf{b}_i \otimes \mathbf{b}_k + \mathbf{b}_k \otimes \mathbf{b}_i). \tag{4.11}$$

In view of (3.23), $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is also a basis of spectral vectors of $\mathbf{J}(t)$ for every $t \in \mathbb{P}$ and we have

$$\mathbf{J}(t)\mathbf{b}_i = \mathrm{e}^{t\gamma_i} \mathbf{b}_i \qquad \text{for all} \quad i \in 3^] \text{ and all t} \in \mathbb{P} \ . \tag{4.12}$$

It follows from (4.11) and (4.12) that

$$\mathbf{J}(t)\mathbf{N}_0\mathbf{J}(t)^\top = \sum_{(i,k) \in 3^] \bigcirc 3^]} \frac{1}{2} \mathrm{e}^{t(\gamma_i + \gamma_k)} N_{i,k} (\mathbf{b}_i \otimes \mathbf{b}_k + \mathbf{b}_k \otimes \mathbf{b}_i) \ ,$$

which, separating the $(1, 1)$-term, may be rewritten as

$$\mathbf{J}(t)\mathbf{N}_0\mathbf{J}(t)^\top = \mathrm{e}^{2t\beta} \Big(N_{1,1} \mathbf{b} \otimes \mathbf{b} + \sum_{(i,k) \in 3^] \bigcirc 3^] \ \setminus \{(1,1)\}} \frac{1}{2} \mathrm{e}^{-t(2\beta - \gamma_i - \gamma_k)} N_{i,k} (\mathbf{b}_i \otimes \mathbf{b}_k + \mathbf{b}_k \otimes \mathbf{b}_i) \Big) \ . \tag{4.13}$$

Since $2\beta - \gamma_i - \gamma_k > 0$ for all $(i, k) \in 3^] \bigcirc 3^]$ with $(i, k) \neq (1, 1)$,we conclude that

$$\lim_{t \to \infty} \mathrm{e}^{-2t\beta} (\mathbf{J}(t)\mathbf{N}_0\mathbf{J}(t)^\top) = N_{1,1} \mathbf{b} \otimes \mathbf{b} \tag{4.14}$$

and hence

$$\lim_{t \to \infty} \mathrm{e}^{-2t\beta} \ \mathrm{tr}(\mathbf{J}(\mathrm{t})\mathbf{N}_0\mathbf{J}(\mathrm{t})^\top) = \mathrm{N}_{1,1} \mathbf{b} \bullet \mathbf{b} = \mathrm{N}_{1,1} \ . \tag{4.15}$$

Assume now that $N_{1,1} \neq 0$. In view of (3.22), it follows from (4.14) and (4.15) that

$$\lim_{t \to \infty} \bar{\mathbf{K}}(t) = \mathbf{b} \otimes \mathbf{b} \ .$$

Now, the set of all $\mathbf{N}_0 \in \mathrm{Sym}_0\mathcal{T}$ such that $N_{1.1} \neq 0$ is an open dense set in $\mathrm{Sym}_0\mathcal{T}$ and hence a neighborhood of $\mathbf{b} \otimes \mathbf{b}$. Therefore $(\mathbf{G}, \mathbf{E}^\top + \mathbf{E}, \mathbf{b} \otimes \mathbf{b})$ is a stable monotonous state. The set of all $\mathbf{N}_0 \in \mathrm{Sym}_0\mathcal{T}$ such that $N_{1.1} = 0$ has an empty interior and hence cannot include the neighborhood of any member of $\mathrm{Sym}_0\mathcal{T}$. Hence $(\mathbf{G}, \mathbf{E}^\top + \mathbf{E}, \mathbf{b} \otimes \mathbf{b})$ is the only monotonous state of exponent $\mathbf{E}$. ∎

Remark 6: If the multiplicity of the largest spectral value is stricly greater than 1, all monotonous states are unstable. ∎

Remark 7: If the assumption that $\mathbf{C}$ is diagonable is dropped, then Prop.9 may still be valid in some cases. In others, there will be no stable monotonous states at all. ∎

4. Viscometric functions.

In this section we consider only the special model. We assume that a state $\sigma := (\mathbf{G}, \mathbf{H}, \mathbf{N})$ of simple shearing in the sense of Def.5 in [N4] is given. As in the previous section, we use $\mathbf{G}$ to endow $\mathcal{T}$ with the structure of an inner-product space. In view or Prop.7 we then have

$$\mathbf{H} = \mathbf{E}^\top + \mathbf{E} \quad \text{with} \quad \mathbf{E}^2 = 0 \quad \text{but} \quad \mathbf{E} \neq 0 \ . \tag{5.1}$$

Making use of Prop.7 in Sect.3 of [4], we choose an orthonormal basis $\mathbf{e} := (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathcal{T}$ such that the matrix of $\mathbf{E}$ relative to $\mathbf{e}$ is of the form

$$[\mathbf{E}] := \kappa \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ , \quad \text{where} \quad \kappa \in \mathbb{P}^\times \ . \tag{5.2}$$

Here and from now on we denote the matrix of any $\mathbf{A} \in \mathrm{Lin}\,\mathcal{T}$ relative to the basis $\mathbf{e}$ by $[\mathbf{A}]$. By (5.2) and (4.5) we have

$$[\mathbf{C}] := \kappa \begin{bmatrix} 0 & \mu & 0 \\ \mu+1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ . \tag{5.3}$$

We consider only the case when

$$\mu > 0 \quad \text{or} \quad \mu < -1 \ . \tag{5.4}$$

An easy calculation shows that the spectrum of $\mathbf{C}$ is

$$\mathrm{Spec}\,\mathbf{C} = \{\kappa\gamma,\ 0, -\kappa\gamma\} \quad \text{with} \quad \gamma := \sqrt{\mu(\mu+1)} \ . \tag{5.5}$$

It follows that $\mathbf{C}$ is diagonable and that $\kappa\gamma$ is its largest spectral value, which has multiplicity 1. Hence Prop. 9 applies and yields:

Proposition 10. *The state* $(\mathbf{G}, \mathbf{H}, \mathbf{N})$ *is a state of simple shearing if and only if one can determine* $\kappa \in \mathbb{P}^{\times}$ *and an orthonormal basis* $\mathbf{e}$ *of* $\mathcal{T}$ *relative to* $\mathbf{G}$ *such that* $\mathbf{H} = \kappa \mathbf{H}_1$ *and*

$$[\mathbf{H}_1] := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad [\mathbf{N}] := \begin{bmatrix} \alpha^2 & \alpha\beta & 0 \\ \alpha\beta & \beta^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \tag{5.6}$$

where

$$\alpha := \sqrt{\frac{\mu}{2\mu + 1}} \quad , \quad \beta := \sqrt{\frac{\mu + 1}{2\mu + 1}} \ . \tag{5.7}$$

Proof : $(5.6)_1$ follows from (5.1) and (5.2). An easy calculation shows that a unit spectral vector $\mathbf{b}$ for the spectral value $\kappa\gamma$ has the component-column

$$[\mathbf{b}\ ] := \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix} \tag{5.8}$$

and hence, since $\mathbf{H} = \mathbf{b} \otimes \mathbf{b}$ by Prop.9, it follows that $(5.6)_2$ holds. ∎

We now use the considerations of Sect.4 of [4] to determine the viscometric functions for a material element described by the special model.

We assume that a configuration $\mathbf{G}$ is given and, as before, use it to endow $\mathcal{T}$ with the structure of an inner product space. Let an orthonormal basis $\mathbf{e}$ of $\mathcal{T}$ be given. Again, we denote the matrix of any $\mathbf{A} \in \operatorname{Lin} \mathcal{T}$ relative to the basis $\mathbf{e}$ by $[\mathbf{A}]$. Let $\kappa \in \mathbb{P}^{\times}$ be given and let $\mathbf{H}_1$ and $\mathbf{N}$ be the lineons whose matrices relative to $\mathbf{e}$ have the form (5.6). Then, by Prop.9, the state

$$\omega_\kappa := (\mathbf{G}, \kappa \mathbf{H}_1, \mathbf{N}) \tag{5.9}$$

is a state of simple shearing. Hence the Assumption I in Sect.4 of [4] is satisfied. Obviously, ω_κ depends continuously on κ and converges to $(\mathbf{G}, \mathbf{0}, \mathbf{N})$ as κ goes to 0. Hence the Assumption II in Sect.4 of [4] is also satisfied.

In view of (1.6) and (5.9), the intrinsic stress $\mathbf{S}_\kappa := \hat{\mathbf{S}}\ (\omega_\kappa)$ produced by the state ω_κ satisfies $\mathbf{S}_\kappa \mathbf{G} = \hat{\mathbf{T}}(\kappa \mathbf{H}_1, \mathbf{N})$, which, by (2.1), gives

$$\mathbf{S}_\kappa \mathbf{G} = \lambda_1 \mathbf{N} + \kappa \left(\frac{\lambda_3}{2} \mathbf{H}_1 + + \frac{\lambda_2}{2} \operatorname{tr}(\mathbf{H}_1 \mathbf{N}) \mathbf{N} + \frac{\lambda_4}{2} (\mathbf{H}_1 \mathbf{N} + \mathbf{N} \mathbf{H}_1) \right) \ . \tag{5.10}$$

Since the matrices of $\mathbf{H}_1$ and $\mathbf{N}$ are given by (5.6) and (5.7), the equation (5.10) can be used to calculate the matrix of $\mathbf{S}_\kappa \mathbf{G}$. The result is

$$[\mathbf{S}_\kappa \mathbf{G}] = \begin{bmatrix} \sigma_1(\kappa) & \tau(\kappa) & 0 \\ \tau(\kappa) & \sigma_2(\kappa) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \tag{5.11}$$

where

$$\tau(\kappa) = \lambda_1\alpha\beta + \kappa\big(\frac{\lambda_3}{2} + \lambda_2\alpha^2\beta^2 + \frac{\lambda_4}{2}\big) \ , \tag{5.12}$$

$$\sigma_1(\kappa) = \lambda_1\alpha^2 + \kappa\alpha\beta\big(\lambda_2\alpha^2 + \lambda_4\big) \ , \tag{5.13}$$

$$\sigma_2(\kappa) = \lambda_1\beta^2 + \kappa\alpha\beta\big(\lambda_2\beta^2 + \lambda_4\big) \ . \tag{5.14}$$

Comparing (5.11) with (4.15) and (4.16) in [4], we see that (5.12), (5.13), and (5.14) are indeed the viscometric functions for the special model of nematic semi-liquids. Hence they can be used to analyse the behavior of nematic semi-liquids for all viscometric flows and, in particular, for Poiseuille flow and Couette flow as in Sects. 6 A and 6 B of [4]. The shear stress function τ given by (5.12) is of the form (6.20) in Remark 6 in [4] with

$$\tau_0 = \lambda_1\alpha\beta \ \text{ and } \ \eta_0 = \frac{\lambda_3}{2} + \lambda_2\alpha^2\beta^2 + \frac{\lambda_4}{2} \ . \tag{5.15}$$

The vector $\mathbf{e}_1$ always gives the flow-direction in viscometric flows. By (5.8), α is the inner product of $\mathbf{b}$ with $\mathbf{e}_1$ and hence the cosine of the angle between the flow direction and the alignment direction.

Remark 8: The equations (5.12), (5.13), and (5.14) reproduce J.L.Ericksen results for his anisotropic fluids, as given by (129.6) in [NLFT]. (There is a small error in (129.6), λ_1 in the first line should be replaced by λ_2.) ∎

Remark 9: Recall that we have assumed that the material constant μ of (2.2) satisfies (5.4). The case when $-1 \geq \mu \geq 0$ remains to be investigated. In this case, I conjecture that there is no state of simple shearing and, if the material element is subjected to a simple shearing process $\mathbf{P}$, the familiy $(\tau_t \mid t \in \mathbb{P})$ defined by (2.4) in [N4] will become asymptotically periodic rather than converge. ∎

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