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by

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Abstract

This technical report considers the design of optimal trajectories of linearly constrained linear quadratic (LQ) systems. It is shown that by applying a Fourier-based state parameterization approach a linearly constrained LQ problem can be converted into a quadratic programming problem. Simulation results show that the proposed approach is an accurate and computationally efficient design tool for determining the optimal solution of linearly constrained LQ problems.

Introduction

Methods for the solution of optimal control problems are well covered in many textbooks {*e.g.,* Athans and Falb, 1966; Kirk, 1970; Sage, 1977; Lewis 1986). Typically, the necessary condition of optimality for a constrained optimal control problem is formulated as a two-point boundary-value problem (TPBVP) using Pontryagin's minimum principle. However, the solution for such a TPBVP is usually difficult, and in some cases not practical, to obtain. In general, variational methods such as Pontryagin's minimum principle are not effective for solving constrained optimal control problems.

In contrast to variational methods, trajectory parameterization approaches offer an alternative means for solving optimal control problems. In general, these techniques convert an optimal control problem into a mathematical programming (MP) problem where a near optimal solution can be obtained via various well developed numerical algorithms. Studies of the relationship between MP and optimal control theory are found in (Canon, Cullum and Polak, 1970; Tabak, 1970; Tabak and Kuo, 1971; Luenberger, 1972; Kraft, 1980; Evtushenko, 1985).

Based on the idea of state parameterization, Nagurka and Yen (1989) developed a Fourier-based method that converts a general optimal control problem into a nonlinear programming (NP) problem. Unlike previous trajectory parameterization algorithms which parameterize control variables, the Fourier-based approach approximates each state variable by a Fourier-type series superimposed on a polynomial. Due to the inverse dynamic nature of the state parameterization approach, the Fourier-based approach does not require integration of the state equations and is thus usually more efficient than control parameterization approaches. Another advantage of the Fourier-based state parameterization method is its ability in handling problems with fixed final states.

The Fourier-based approach was specialized by Yen and Nagurka (1988, 1989) to solve unconstrained time-invariant LQ problems where the condition of optimality is formulated as a system of linear algebraic equations. Simulation results show that the Fourier-based approach is more efficient than standard LQ problem solvers in handling high order systems. This report further demonstrates the utility of the Fourier-based approach and extends it by developing a computational tool for the solution of linearly constrained optimal control problems. In particular, the Fourier-based approach is applied to convert linearly constrained LQ problems into linear constrained mathematical programming problems which can be solved by well developed routines.

In this report the following mathematical notation is employed. Scalar quantities (values) are denoted by plain lower case letters. Scalar variables are denoted by italic lower case letters. Vectors are denoted by boldface lower case letters. Boldface upper case letters are used to represent matrices. The only exception is Y which is used to represent the state parameter vector. The superscript T denotes the transpose of a vector or matrix. Vectors are assumed to be column vectors by default. Matrix inverses are denoted in the usual way by superscript -1. The inverse transpose is denoted by superscript -T.

Problem Statement

The derivation that follows considers linear systems described by the state space model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \qquad (1)$$

with known initial condition $x(0) = x_0$ where x is an *N x 1* state vector, **u** is an */ x 1* control vector, *AisanNxN* system matrix, and B is an*Nx J* control matrix. For now, it is assumed that / = *N, i.e.,* the number of control variables is equal to the number of state variables. (The case *J* <*N* will be addressed later). Furthermore, it is assumed that the control matrix B is invertible. As a result, every state variable can be "actively" controlled.

The design goal is to find the optimal control *u(t)* and the corresponding state trajectory x(f) in the time interval [0,7] that minimizes the quadratic performance index, L,

$$L = L\backslash + Li \qquad (2)$$

where

$$Li = x^T H x + h^T x \qquad (3)$$

$$L_2 = \int_0^T ([x^T Q x + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{x}^T\mathbf{P}\mathbf{u} + \mathbf{a}^T\mathbf{x} + \mathbf{b}^T\mathbf{u})\, dt \qquad (4)$$

without violating the linear inequality constraints:

Ei(r)xU) + E2(f)u(f) ^ e(r) (5)

In this report, superscript T denotes transpose and *T* (italic) represents the final time which is assumed known.

Fourier-Based State Parameterization

This section describes the basic idea of the multiple-segment Fourier-based state parameterization approach. The first step of the approach is to divide [0,71 into / intervals [*o, *h]9* *[tu* '2L • • •» *[*i-i>* *U*1 where $r_0 = 0$ and r/ = *T*. (Later in this report it is shown that for many problems 7 = 1 , *i.e.*, a single segment parameterization is appropriate). In the time interval *[t^Ji]* 0' = 1>2 ,..., *I)* the *n-th* state variable $x_n(t)$ is approximated by the sum of a third-order auxiliary polynominal $d_{in}(t)$ and a *K* term Fourier-type series, *i.e.*, for *i* = 1,..., /, *n* = 1,..., *N*,

$$x_n(t) \;=\; <*«« + \overset{K}{Z}, ***cos\left(\frac{2k\pi\,(t-t_{i-1})}{\Delta t_i}\right) + \sum_{k=1}^{K} b_{ink}\, sin\left(\frac{2k\pi\,(t-t_{i-1})}{\Delta t_i}\right) \qquad (6)$$

where

$$d_{in}(t) = d_{ino} + d_{in1}\,(t-t_{i-1}) + d_{in2}\,(t-t_{i-1})^2 + d_{jn3}\,(t-t_{i-1})^3 \qquad (7)$$

$$\Delta t_i \;=\; t_i - t_{i-1} \qquad (8)$$

Compared to a standard Fourier series expansion, this representation assures high speed of convergence and differentiability as described in (Nagurka and Yen, 1989).

If Xi_{no}, $\dot{x}ino$, $x_{in}r$, and $\dot{x}_{in}T$ are the values of the state variable x_n and its derivative at the boundaries of the time segment *[t^Ji]*, *i.e.*9

$$*ino \;=\; Xn(ti-\backslash) \qquad \dot{*}ino=\dot{X}_n(ti-i) \qquad *inT \;-\; X_n(U) \qquad \dot{X}inT \;=\; \dot{X}_n(U) \qquad (9d\text{-}d)$$

then the four coefficients of the auxiliary polynominal ^(r) can be written as functions of the boundary values of the segment *[tu,tj* and the coefficients of the Fourier series, *i.e.*t

$$d_{ino} = *ino \;"\; \sum_{k=1}^{K} jLGink \;\;>\;\; d'_{in}\backslash = \dot{X}\wedge\frac{2\wedge}{A^{t}i}\sum_{k=1}^{K} X\;{}^{k\;b}ink \qquad (10a,l0b)$$

$$din! \;=\; 3(x_{inT} \;-\; Xino \;+\; 4^* \sum_{k=1}^{K} * \;\; *mJk)\;\; A//^2 \;-\; 2(\dot{x}_{ino} \;+\; \dot{X}\wedge j)\;\; A\wedge^1 \qquad (10c)$$

$$d_{in3} \;=\; 2(x_{inT} \;-\; x_{ino} + 2\pi \sum_{k=1}^{K} k\; b_{ink})\;\Delta t_i^{-3} + (\dot{x}_{ino} + \dot{x}_{inT})\;\Delta t_i^{-2} \qquad (10d)$$

Substituting these expressions into equation (6) gives

$$x_n(t) = Pil*ino + \rho_{i2}\dot{x}_{ino} + \rho_{i3}x_{inT} + \rho_{i4}\dot{x}_{inT} + \sum_{*=1}^{K} \left(\alpha_{ik}\, a_{ink} + \beta_{ik}\, b_{ink}\right) \tag{11}$$

where

$$\rho_{i1} = 1 - 3\tau_i^2 + 2\tau_i^3 \quad , \quad \rho_{i2} = \left(\tau_i - 2\tau_i^2 + \tau_i^3\right)\Delta t_i \tag{12a,b}$$

$$\rho_{i3} = 3\tau_i^2 - 2\tau_i^3 \quad , \quad \rho_{i4} = \left(- \tau_i^2 + \tau_i^3\right)\Delta t_i \tag{12c,d}$$

$$\alpha_{ik} = \cos\ (2**7;) -1 \quad _f\ A* = \sin\left(2k\pi\tau_i\right) - 2k\pi\tau_i\left(1 - 3\tau_i + 2\tau_i^2\right) \tag{12e,f}$$

with

$$\tau_i = (t - t_{i-1})/\Delta t_i \tag{13}$$

The terms p,-i,...f p/4, crjk, *pit* are functions of time *t*. Equation (11) can be written in compact form as

$$x_n it) = p7(0Jin \qquad for \quad t_{iA} < t < ti \tag{14}$$

where

$$\mathbf{Pi}^T(0 = [\ p_n \quad p_a \quad p_{i3} \quad p_{i4} \quad an \quad \ldots \quad a_{iK} \quad p_{ix} \quad \ldots \quad Ax\ ] \tag{15}$$

$$Jin = [\ Xi\wedge \quad \dot{X}i_{no} \quad Xinj \quad \dot{X}_{in}T \quad d\backslash n\backslash \quad \cdots \quad QMC \quad *inl \quad \cdots \quad ClinK\ Y$$

$$= [y«il \quad Jm2 \quad \cdots \quad y/nAf]^1 \tag{16}$$

are vectors of dimension $M = 4 + 2/if$. Note that the bold face letter p_g is a vector and the italic letter p^ represents a scalar variable. The first four elements of ji_n are the values of x_n and $\dot{x}_n$ at the boundary of $[r_{f\text{-}\cdot lf}\ f/]$; the remaining elements are the coefficients of the Fourier-type series. Vector $y_{\dot{x}n}$ can be viewed as a state parameter vector which characterizes the actual trajectory of x_n over the time interval fa.;, f;]. The design goal is to search for the optimal values of the elements of y^ for i'= 1,..., /, n = 1,..., N_9 such that the performance index is minimized. This goal is achieved by first writing the state vector, its rate and the control vector as functions of the state parameters.

The state vector x(r) can now be written as

$$x(r) = \overline{p};(r)Y_i \qquad for t_{iA} < t < ti \tag{17}$$

where

$$\bar{\rho}_i = \begin{bmatrix} \rho_i^T & & & 0 \\ & \rho_i^T & & \\ & & \ddots & \\ 0 & & & \rho_i^T \end{bmatrix}, \quad Y_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iN} \end{bmatrix} = \begin{bmatrix} [y_{i11} \cdots y_{i1M}]^T \\ [y_{i21} \cdots y_{i2M}]^T \\ \vdots \\ [y_{iN1} \cdots y_{iNM}]^T \end{bmatrix} \qquad (18a,b)$$

By direct differentiation of equation (17), $\dot{x}(t)$ can be written as:

$$\dot{x}(t) = \bar{\sigma}_i(t) Y_i \qquad for\ \ t_{i-1} < t \le t_i \qquad (19)$$

where

$$\bar{\sigma}_i(t) = \dot{\bar{\rho}}_i(t) \qquad (20)$$

Note that since it is assumed that B^{-1} exists, equation (1) can be rewritten as:

$$u = B^{-1}\dot{x} + V x \qquad (21)$$

where

$$V = -B^{-1} A \qquad (22)$$

Substituting equations (17) and (19) into equation (21) gives

$$u = (B^{-1}\bar{\sigma}_i(t) + V\bar{\rho}_i(t))\, Y_i \qquad for\ \ t_{i-1} < t \le t_i \qquad (23)$$

Thus, using the Fourier-based state parameterization approach, all the variables appearing in the state equation (including the state vector, state rate vector, and control vector) can be represented as functions of the state parameter vector. By employing this representation, the LQ problem can be reformulated as a quadratic programming (QP) problems with the state parameters as new variables.

Unconstrained LQ Problems

The first goal of this section is to demonstrate the conversion process from a LQ problem to a QP problem via the Fourier-based state parameterization approach. The second goal is to develop a solution approach for the converted QP problem. It will be shown that the converted QP problem can be formulated as an unconstrained optimization problem with a quadratic objective function.

The first step in the conversion is to rewrite the performance index as a function of state parameter vectors $\mathbf{Y}_i$. The performance index L_1 can be written as a function of $\mathbf{Y}_I$ by noting that the terminal state vector can be represented as:

$$\mathbf{x}(T) = \mathbf{\Theta Y}_I \qquad (24)$$

where $\mathbf{\Theta}$ is a transformation matrix with elements

$$\theta_{nm} = \begin{cases} 1 & m = (n\text{-}1)\,M + 3 \quad for \quad n = 1, ..., N \\ 0 & otherwise \end{cases} \qquad (25)$$

Substituting equation (24) into equation (3) gives

$$L_1 = \mathbf{Y}_I^{\mathrm{T}}(\mathbf{\Theta}^{\mathrm{T}}\mathbf{H\Theta})\mathbf{Y}_I + \mathbf{h}^{\mathrm{T}}\mathbf{\Theta Y}_I \qquad (26)$$

Similarly, the performance index L_2 can be written as a function of $\mathbf{Y}_i$ although the process is somewhat more complicated. Substituting equation (21) into the integrand of equation (4) gives:

$$\mathbf{x}^{\mathrm{T}}\mathbf{Qx} + \mathbf{u}^{\mathrm{T}}\mathbf{Ru} + \mathbf{x}^{\mathrm{T}}\mathbf{Pu} + \mathbf{a}^{\mathrm{T}}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{u} = \mathbf{x}^{\mathrm{T}}\mathbf{F}_1\mathbf{x} + \dot{\mathbf{x}}^{\mathrm{T}}\mathbf{F}_2\dot{\mathbf{x}} + \dot{\mathbf{x}}^{\mathrm{T}}\mathbf{F}_3\mathbf{x} + \mathbf{c}_1\mathbf{x} + \mathbf{c}_2\dot{\mathbf{x}} \qquad (27)$$

where

$$\mathbf{F}_1 = \mathbf{Q} + \mathbf{V}^{\mathrm{T}}\mathbf{RV} + \mathbf{PV} \qquad (28)$$

$$\mathbf{F}_2 = \mathbf{B}^{-\mathrm{T}}\mathbf{RB}^{-1} \qquad (29)$$

$$\mathbf{F}_3 = 2\mathbf{B}^{-\mathrm{T}}\mathbf{RV} + \mathbf{B}^{-\mathrm{T}}\mathbf{P} \qquad (30)$$

$$\mathbf{c}_1 = \mathbf{a} + \mathbf{V}^{\mathrm{T}}\mathbf{b} \qquad (31)$$

$$\mathbf{c}_2 = \mathbf{B}^{-\mathrm{T}}\mathbf{b} \qquad (32)$$

By substituting equations (17) and (19) into equation (27), the integrand of the performance index can be expressed as a function of parameter vector $\mathbf{Y}_i$ such that

$$\mathbf{x}^{\mathrm{T}}\mathbf{Qx} + \mathbf{u}^{\mathrm{T}}\mathbf{Ru} + \mathbf{x}^{\mathrm{T}}\mathbf{Pu} + \mathbf{a}^{\mathrm{T}}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{u} = \mathbf{Y}_i^{\mathrm{T}}\mathbf{\Lambda}_i\mathbf{Y}_i + \mathbf{Y}_i^{\mathrm{T}}\mathbf{\Gamma}_i \quad for \quad t_{i-1} \le t \le t_i \qquad (33)$$

where

$$\mathbf{\Lambda}_i = \rho_i\rho_i^{\mathrm{T}} \otimes \mathbf{F}_1 + \sigma_i\sigma_i^{\mathrm{T}} \otimes \mathbf{F}_2 + \sigma_i\rho_i^{\mathrm{T}} \otimes \mathbf{F}_3 \qquad (34)$$

$$\mathbf{\Gamma}_i = \rho_i \otimes \mathbf{c}_1 + \sigma_i \otimes \mathbf{c}_2 \qquad (35)$$

In equations (34) and (35), $\otimes$ is a Kronecker product sign. Note that the elements of ρ_i and σ_i are functions of time t and time interval Δt_i.

Using the results of equation (33), the integral part of the performance index can be expressed as

$$J_2 = \sum_{i=1}^{I} \int_{t_{i-1}}^{t_i} \left(\mathbf{Y}_i^T \Lambda_i \mathbf{Y}_i + \mathbf{Y}_i^T \Gamma_i \right) dt = \sum_{i=1}^{I} \mathbf{Y}_i^T \mathring{\Lambda}_i \mathbf{Y}_i + \mathbf{Y}_i^T \mathring{\Gamma}_i \tag{36}$$

where

$$\mathring{\Lambda}_i = \int \Lambda_i dt \qquad \mathring{\Gamma}_i = \int \Gamma_i dt \tag{37),(38}$$

For time-invariant problems, upon substituting equations (34) and (35) into equation (36), F_1, F_2, F_3, C_1, and C_2 can be removed from the integral, and the remaining integral part of Λ_i and Γ_i can be evaluated analytically. These evaluations have been summarized in tables for the integrals of elements of p_i and a_i and the products (and cross-products) of the elements of p_i and Θ_i. The availability of such integral tables makes the approach numerically integration-free in handling time-invariant problems.

By substituting equations (26) and (36) into equation (2), the performance index can be written as a quadratic function:

$$L = \sum_{i=1}^{I-1} \left(\mathbf{Y}_i^T \mathring{\Lambda}_i \mathbf{Y}_i + \mathbf{Y}_i^T \mathring{\Gamma}_i \right) + \mathbf{Y}_I^T \left(\Theta^T \mathbf{H} \Theta + \mathring{\Lambda}_I \right) \mathbf{Y}_I + \mathbf{Y}_I^T \left(\mathring{\Gamma}_I + \Theta^T \mathbf{h} \right) \tag{39}$$

Equation (39) can be put into a more compact form as

$$L = \hat{\mathbf{Y}}^T \hat{\Omega} \hat{\mathbf{Y}} + \hat{\mathbf{Y}}^T \hat{\omega} \tag{40}$$

where

$$\hat{\mathbf{Y}}^T = \begin{bmatrix} \mathbf{Y}_1^T & \mathbf{Y}_2^T & \cdots & \mathbf{Y}_I^T \end{bmatrix} \tag{41}$$

$$\hat{\Omega} = \begin{bmatrix} \mathring{\Lambda}_1 & & & & \\ & \mathring{\Lambda}_2 & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \mathring{\Lambda}_{I-1} & \\ & & & & \mathring{\Lambda}_I + \Theta^T \mathbf{H} \Theta \end{bmatrix}, \quad \hat{\omega} = \begin{bmatrix} \mathring{\Gamma}_1 \\ \mathring{\Gamma}_2 \\ \vdots \\ \mathring{\Gamma}_{I-1} \\ \mathring{\Gamma}_I + \Theta^T \mathbf{h} \end{bmatrix} \tag{42),(43}$$

In minimizing this converted performance index, there are two types of constraints that must be satisfied. The first set of constraints refers to the given initial conditions and can be expressed as:

yini = Xno *for* **n = 1 ,..., *N*** **(44)**

where x«> is the initial value of the state variable x_n. The second set of constraints refers to the continuity requirements. That is, to ensure continuity between segments it is required that:

(i-l)nT* = X/no » **(i-l)nT* = *mo , *far* 1 = 1, ..., / , *n* = 1, ..., *N* **(45)

These equations are equivalent to

y<i-l)n3 = *yinl* , *y(i-l)n4* = *Tml* > *far* *I* = 1, ..., / , *n* = 1,..., *N* **(46)**

The optimization problem can now be formulated as the search for ;y$_{lnm}$,, i =1,..., /, *n* = 1,..., *N,m* = 1,.~, *M,* that minimizes the performance index of equation (40) subject to the equality constraints of equations (44) and (46).

The goal of the following part of this section is to develop a solution approach for this equality constrained QP problem by converting it into an unconstrained QP problem. To accomplish this goal, a new state parameter vector z is introduced, specified as

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_4 \end{bmatrix}$$ <«>

where

$z_1 = [\ x_{1T}^T \quad \dot{x}_{1T}^T$ •••*?,,,,• *(V)$]^T$ **W**

$z_2 = [\ x_{2o}^T \quad \dot{x}_{2o}^T$ ·{ *]Y **(49)**

$z_3 = [\ a_1^T \quad b_1^T \quad \cdots \quad a_7^T \quad b_7^T \quad \dot{x}_{Xo}^T \quad \dot{x}_{IT}^T \quad XJJ^T\ ]^T$ **(50)**

with

$\mathbf{x}_{io}$ = *%i\o %i2,o* " * * ^*iNo* J ·\^~/

X/^ — [^ *,'i\o *%i2o* * *" *Xfflo* J ^^^\ W^/

$\mathbf{x}_{iT}$' = [*iir 0:/27 •" • **iNT* $]^T$ **(54)**

$$\dot{\mathbf{x}}_{iT} = [\ \dot{x}_{i1T}\ \ \dot{x}_{i2T}\ \cdots\ \dot{x}_{iNT}\ ]^{T} \qquad (55)$$

$$a_{j} = [\ a_{i11}\ \cdots\ a_{i1K}\ \ a_{i21}\ \cdots\ a_{i2K}\ \cdots\cdots\ a_{iN1}\ \cdots\ a_{iNK}\ ]^{T} \qquad (56)$$

$$b, = [\ \text{friii}\ \cdots\ {}^{b}HK\ \ \text{fr/21}\ \cdots\ b_{i2K}\ \cdots\cdots\ b_{iN1}\ \cdots\ {}^{b}iNK\ ]^{T} \qquad (57)$$

Physically, z\ is a vector of the values of the state and state rate vectors at the beginning of all but the first segment. Similarly, Z2 is a vector of the values of the state and state rate vectors at the end of all but the last segment. The first part of Z3 is a vector of the Fourier coefficients; the second part of Z3 is a vector of the unknown boundary values of the state and state rate vectors at the boundaries of [0,71. In contrast, Z4 is a vector of the known boundary values of the state and state rate vectors, i.e., in this case it is the initial value of the state vector. From the definitions of z\ and Z2, it is clear that the continuity requirement of equations (45) and (46) can be satisfied by equating z\ = Z2.

Based on the definitions of z and Y, a linear transformation relation between these two vectors can be established as

$$\hat{\mathbf{Y}} = \mathbf{W}\mathbf{z} \qquad (58)$$

The performance index of equation (40) can thus be rewritten as a function of z

$$L = \mathbf{z}^{T}\text{£}2\mathbf{z} + \mathbf{z}^{T}\text{co} \qquad (59)$$

where

$$\mathbf{Q} = \mathbf{W}^{T}\mathbf{S}\,\mathbf{w} \qquad (60)$$

$$\text{co} = \mathbf{W}^{T}\mathbf{G} \qquad (61)$$

Using the definition of z from equation (47), the performance index of equation (59) can be expressed as

$$L = \left[\mathbf{z}_1^T\ \mathbf{z}_2^T\ \mathbf{z}_3^T\ \mathbf{z}_4^T\right]\begin{bmatrix} \Omega_{11} & \text{Ql2} & \text{O13} & \text{Ql4} \\ \Omega_{21} & \text{G22} & \text{G23} & \text{G24} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} \\ \Omega_{41} & \text{Q42} & \text{"43} & \text{"44} \end{bmatrix}\begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} + \left[\mathbf{z}_1^T\ \mathbf{z}_2^T\ \mathbf{z}_3^T\ \mathbf{z}_4^T\right]\begin{bmatrix} \omega_1 \\ \text{©2} \\ \text{0>3} \\ \text{©4} \end{bmatrix} \qquad (62)$$

Since z\ = $\mathbf{z}_2$, equation (62) can be collapsed to

$$L = \left[\mathbf{z}_1^T\ \mathbf{z}_3^T\ \mathbf{z}_4^T\right]\begin{bmatrix} \Omega_{11}+\Omega_{22}+\Omega_{12}+\Omega_{21} & \Omega_{13}+\Omega_{23} & \Omega_{14}+\Omega_{24} \\ \Omega_{31}+\Omega_{32} & \text{"33} & \text{Q34} \\ \text{Q41+Q42} & \text{Q43} & \text{Q44} \end{bmatrix}\begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix}$$

$$+\left[z_1^T\, z_3^T\, z_4^T\right]\begin{bmatrix} 0)1+0)2 \\ «\% \\ 0)_4 \end{bmatrix} \qquad (63)$$

Decoupling the known vector Z4 from the rest of unknown part of the state parameter vector gives

L = $\hat{Z}^T$X11$\hat{Z}$ + $\hat{Z}^1$\(A*i2 + A,21)Z4 + T|' +/Z (54)

where

$$\hat{z} = \begin{bmatrix} z_1 \\ 23 \end{bmatrix} \qquad (65)$$

$$\lambda_{11} = \begin{bmatrix} \Omega_{11}+\Omega_{22}+\Omega_{12}+\Omega_{21} & \Omega_{13}+\Omega_{23} \\ \Omega_{31}+\Omega_{32} & "33 \end{bmatrix} \qquad (66)$$

$$\lambda_{12} = \begin{bmatrix} \Omega_{14}+\Omega_{24} \\ \Omega_{34} \end{bmatrix} \qquad (67)$$

$$\lambda_{21} = \left[\ \Omega_{41}+\Omega_{42} \quad \Omega_{34}\ \right] \qquad (68)$$

$$\eta = \begin{bmatrix} ©1+CD2 \\ ©3 \end{bmatrix} \qquad (69)$$

$$\mu = z_4^T \text{\^{}2224} + z_4^T C04 \qquad (70)$$

Equation (64) is a quadratic function of the unknown part of the state parameter vector $\hat{z}$. For an unconstrained LQ problem, the necessary condition of optimal solution can be obtained by differentiating the performance index with respect to $\hat{z}$. This leads to

$$(k_n+\backslash^T n)z\hat{}= -(\lambda_{12}+\lambda_{21}^T)z_4 - r| \qquad (71)$$

from which the unknown part of the state parameter vector, $\hat{z}$, can be solved.

The same solution procedure can also be applied to problems with fixed terminal states. The only modification required is to redefine Z3 and Z4 as

$$z_3 = [\ aj \quad b_1^T \quad \cdots \quad a_l^T \quad bj \quad \dot{x}j_o \quad \dot{x}_{lT}^T\ ]^T \qquad (72)$$

$$z_4 = [\ x_{1T}^T \quad x_{1o}^T\ ]^F \qquad (73)$$

since the terminal value of the state vector is known. Similarly, problems with fixed initial and/or final state rate vectors can also be handled by this approach.

Linearly Constrained LQ Problems

The goal of this section is to develop the conversion process from a linearly constrained LQ problem to a QP problem using the Fourier-based state parameterization approach. In particular, the state space inequality constraints of equation (5) can be converted into a system of linear algebraic constraints.

Recall the inequality constraints of equation (5)

Ei(f)x(0 + E2(0u(0≤ e(f) (74)

Substituting equation (21) into the above equation gives

Si(r)x(r) + S2(0*(r) ^ e(r) (75)

where

$$\mathbf{S}_1 = \mathbf{E}_1 + \mathbf{E}_2\mathbf{V} \quad (76)$$

$$\mathbf{S}_2 = \mathbf{E}_2\mathbf{B}^{-1} \quad (77)$$

Substituting equations (17) and (19) into equation (75) gives

{Si(t)pi(t) + S_2(t)Oi(t)]Yi = G,<f)Y,-£ e(r) *for ti.\<t≤ti* –, i = 1,..,/ (78)

where

G,(r) = Si(r)pi(r) + S_2(r)aK0 *for* r_M <≥ *t*≤ *a* , / = 1,..,/ (79)

Note that the constraints of equation (78) are functions of time, a continuous variable. Consequently, equation (78) represents an infinite number of constraints which need to be satisfied along the trajectory. In order to convert these constraints into a finite number of algebraic inequalities, these constraints are relaxed to be satisfied only at a finite number of points (usually chosen to be equally spaced) in time. Consequently, equation (78) is replaced by

ĜiYiZ ê, *for* / = 1, . . . , / (80)

where

$$\hat{\mathbf{G}}_i = \begin{bmatrix} \mathbf{G}_i(t_{i-1}) \\ \mathbf{G}_i(t_{i-1} + \delta t_i) \\ \vdots \\ \mathbf{G}_i(t_{i-1} + (p_i-1)\delta t_i) \\ \mathbf{G}_i(t_i) \end{bmatrix}, \quad \hat{\mathbf{e}}_i = \begin{bmatrix} \mathbf{e}_i(t_{i-1}) \\ \mathbf{e}_i(t_{i-1} + \delta t_i) \\ \vdots \\ \mathbf{e}_i(t_{i-1} + (p_i-1)\delta t_i) \\ \mathbf{e}_i(t_i) \end{bmatrix} \quad (81),(82)$$

with

$$\delta t_i = \frac{\Delta t_i}{p_i} \tag{83}$$

where p_i is the number of sampling points for the j-th segment. Equation (80) can be put into the following compact from

$$G\hat{Y} \le e^* \tag{84}$$

where

$$G = \begin{bmatrix} \hat{G}_1 & & & 0 \\ & \hat{G}_2 & & \\ & & \ddots & \\ 0 & & & \hat{G}_l \end{bmatrix}, \quad e^* = \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_l \end{bmatrix} \tag{85),(86}$$

By using equation (58), these constraints can be rewritten in terms of z. This gives

$$G^* z \le e^* \tag{87}$$

where

$$G^* = GW \tag{88}$$

Similar to equation (62), the inequality constraints of equation (87) can be represented as

$$\begin{bmatrix} G^*_{11} & G^*_{12} & G^*_{13} & G^*_{14} \\ G^*_{21} & G^*_{22} & G^*_{23} & G^*_{24} \\ G^*_{31} & G^*_{32} & G^*_{33} & G^*_{34} \\ G^*_{41} & G^*_{42} & G^*_{43} & G^*_{44} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \le \begin{bmatrix} e^*_1 \\ e^*_2 \\ e^*_3 \\ e^*_4 \end{bmatrix} \tag{89}$$

With $z_1 = z_2$, equation (89) collapses to

$$\begin{bmatrix} G^*_{11}+G^*_{22}+G^*_{12}+G^*_{21} & G^*_{13}+G^*_{23} & G^*_{14}+G^*_{24} \\ G^*_{31}+G^*_{32} & G^*_{33} & G^*_{34} \\ G^*_{41}+G^*_{42} & G^*_{43} & G^*_{44} \end{bmatrix} \begin{bmatrix} z_1 \\ z_3 \\ z_4 \end{bmatrix} \le \begin{bmatrix} e^*_1+e^*_2 \\ e^*_3 \\ e^*_4 \end{bmatrix} \tag{90}$$

Since z_4 is a known vector, the corresponding terms can be moved to the right hand side of the equation. This gives

$$\hat{G}\hat{z} \le \hat{e} \tag{91}$$

where

$$\widehat{G} = \begin{bmatrix} G^*_{11} + G^*_{22} + G^*_{12} + G^*_{21} & G^*_{13} + G^*_{23} \\ G^*_{31} + G^*_{32} & G^*_{33} \\ G^*_{41} + G^*_{42} & G^*_{43} \end{bmatrix}, \quad \mathbf{e}^* = \begin{bmatrix} e^*_1 + e^*_2 - (G^*_{14} + G^*_{24}) z_4 \\ e^*_3 - G^*_{34} z_4 \\ e^*_4 - G^*_{44} z_4 \end{bmatrix} \quad (92),(93)$$

and vector $\widehat{\mathbf{z}}$ is **defined in** equation (65).

In summary, by applying the Fourier-based state parameterization approach, a linearly constrained LQ problem can be converted into a QP problem where a quadratic function of equation (64) is required to be minimized without violating a system of linear algebraic inequalities of equation (91).

Fourier-Based Approach for General Linear Systems

The approach presented above is applicable only for systems with square and invertible control matrices. This section generalizes the Fourier-based approach to the more common case of general linear systems which have fewer control variables than state variables. The system of is again has the linear structure described by equation (1). In this case, the control matrix, B, is an *N x J* matrix where the number of state variables, *N*, is greater than the number of control variables, /. It is assumed that the rank of the control matrix B is equal to /.

To apply the Fourier-based approach, the state equation of equation (1) is first modified as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}'\mathbf{u}'(t) \quad (94)$$

where

$$\mathbf{B}' = \mathbf{B}'_{NxN} = \begin{bmatrix} \mathbf{I}_{(N-J)x1} & \mathbf{B}_{NxJ} \\ \mathbf{O}_{Jx1} & \end{bmatrix} \quad (95)$$

and

$$\mathbf{u}' = \mathbf{u}'_{Nx1} = \begin{bmatrix} \widehat{\mathbf{u}}_{(N-J)x1} \\ \mathbf{u}_{Jx1} \end{bmatrix} \quad (96)$$

with the subscripts representing the dimensions of the matrices. By introducing an artificial control vector, $\widehat{u}$, the new control matrix, B', can be inverted and the Fourier-based approach is thus applicable. In order to predict the optimal solution, the performance index is modified as

$$L' = L + r\int_0^T \left(\hat{\mathbf{u}}^T\hat{\mathbf{u}}\right) dt \tag{97}$$

where L is the performance index of the original LQ problem and r is a weighting constant chosen to be a large positive number. The integral term associated with r is used to represent the contribution of the artificial control.

The advantage of using artificial control variables is that a non-actively controlled system can be converted into an actively controlled system. Consequently, the Fourier-based state parameterization becomes immediately applicable. The trade-off is that the resulting solution will not, in a strict mathematical sense, satisfy the trajectory admissibility requirement (see Yen and Nagurka, 1988) due to the existence of artificial control variables. However, by penalizing the artificial control vector, the magnitude and influence of the artificial control variables can be made insignificant and the solution of the modified optimal control problem can become a near optimal solution of the original LQ problem.

Simulation Studies

For the simulation studies reported here, LQ problems are solved by the Fourier-based approach and compared with closed-form solutions or solutions obtained by standard numerical algorithms. Examples 1 and 2 are designed to the study the effectiveness of the Fourier-based approach in solving unconstrained LQ problems. In particular, Example 1 considers a problem with an actively controlled structure. Example 2 investigates a general linear system. Examples 3 and 4 are used to study the effectiveness of the Fourier-based state parameterization method in handling linearly constrained LQ problems. In particular, Example 3 considers a LQ problem with a linear state constraint and Example 4 examines a problem with a bounded control variable.

To check accuracy, the values of the performance index from standard approaches and the Fourier-based approach are compared. The computer programs used in the simulations were written in the "C" language and compiled by a Turbo C compiler (Version 2.0). Efforts were made to optimize the speed of the computer codes. The simulations were executed on a 16 MHz NEC 386 PowerMate personal computer with a 16 MHz 80387 coprocessor.

For the first two examples, the time (in seconds) required to execute the program was recorded for each simulation and was used as an index of the computational efficiency. For the first two examples, a transition matrix approach was applied to generate the state and control variables at prespecified equally-spaced points in time for unconstrained LQ problems. For the last two examples where the linearly constrained LQ problems are converted into general QP problems, the numerical algorithm developed by (Gill and Murray,1977), which is considered as to be one of the most efficient solution approaches for QP problems, is implemented and applied to determine the optimal value for the unknown state parameters.

Example 1: The goal of this example is to investigate the effectiveness of the Fourier-based approach for solving high order LQ problems for systems with invertible control matrices. Consider an N input N-*th* order system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x}^T(0) = [\,1 \;\; 2 \cdots N\,] \tag{98}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} & 1 & & & \\ & & 1 & & \mathbf{0} \\ & & & \ddots & \\ & \mathbf{0} & & & 1 \\ 1 & -2 & \cdots & & (-1)^N N \end{bmatrix}_{N \times N}, \quad \mathbf{B} = \mathbf{I}_{N \times N} \tag{99}$$

The performance index is

$$L = \mathbf{x}^T(1)\mathbf{H}\mathbf{x}(1) + \int_0^1 (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,dt, \quad \mathbf{H} = 10\mathbf{I}_{N \times N}, \quad \mathbf{Q} = \mathbf{R} = \mathbf{I}_{N \times N} \tag{100}$$

Simulation results for N = 2, 3,..., 16 are summarized in Table I assuming a single-segment, two-term Fourier-based approach (*i.e.* $J = 1$ and $K = 2$). The time histories of the state and control variables of the case of $N = 2$ are plotted in Figures 1a and 1b, respectively. The results demonstrate that a single-segment Fourier-based approximation is accurate (i.e., the error of the performance index value is always less than 1%) *for* all cases studied and is especially efficient in solving optimal control problems for high order systems.

Table I: Summary of Simulation Results of Example 1

	Transition-Matrix Approach		Fourier-Based Approach*		Comparison	
N	Performance Index	Time	Performance Index	Time	%Time*	*A%L*[c]
2	5.3591	0.22	5.3591	0.39	177.3	< 3.7 *x* 10-[5]
3	44.0044	0.44	44.0045	0.66	150.0	<5.9xl(H
4	44.2499	0.87	44.2504	1.05	120.7	< 1.1 *x* 10-3
5	164.3776	1.48	164.3884	1.59	107.4	< 6.6x10-3
6	153.7563	2.36	153.7622	2.37	100.4	< 3.9 *x* 10-3
7	399.9883	[illegible].40	400.1103	3.29	96.8	< 3.1 *x* 10-2
8	373.0219	5.16	373.0597	4.56	88.4	< 1.1 *x* 10-2
9	788.1612	7.15	788.8568	6.04	84.5	< 8.9x10-2
10	741.6136	9.51	741.7737	7.85	82.5	< 2.2x10-2
11	1366.9437	12.96	1369.5209	9.99	77.1	<1.9xlO-[1]
12	1299.3828	16.64	1299.8946	12.58	75.6	< 3.6x10-2
13	2175.1952	20.81	2182.3431	15.44	74.2	< 3.3 x 10-[1]
14	2086.3916	26.91	2087.7219	18.73	69.6	< 6.4xlO-[2]
15	3252.2758	32.62	3268.4011	22.52	69.0	<5.0x10-[1]
16	3142.8478	41.08	3145.8080	26.97	65.7	< 9.5 x 10-2

"With single segment two-term Fourier-type series

^Percent of execution time of Fourier-based approach relative to execution time of transition-matrix approach

Percent difference of performance index of Fourier-based approach relative to performance index value of transition-matrix approach

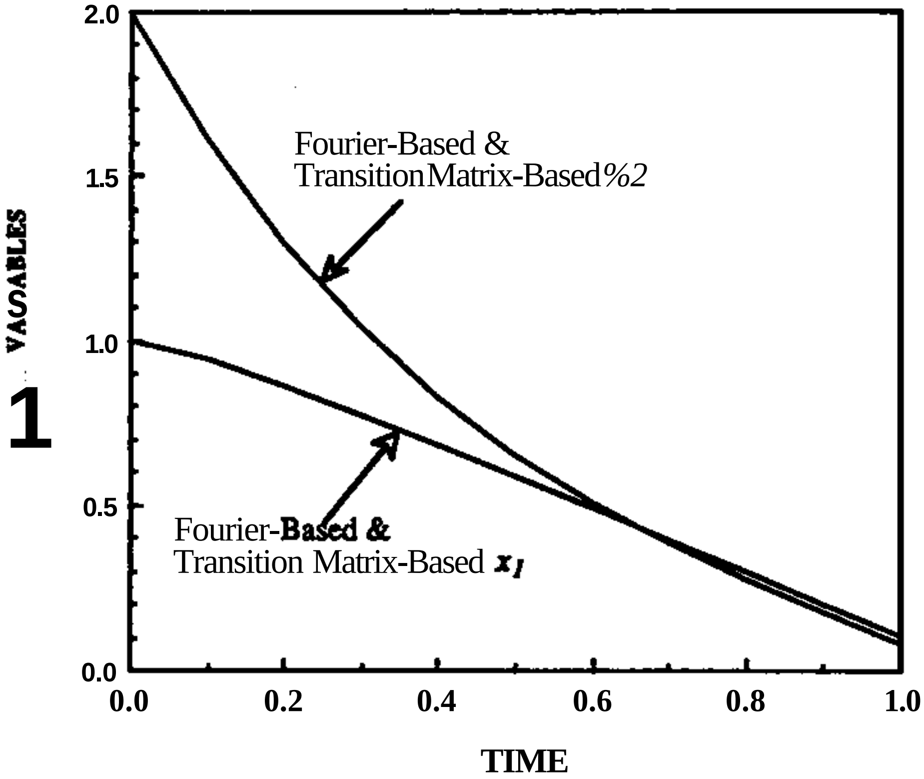

Figure la. State Variable Histories for Example 1

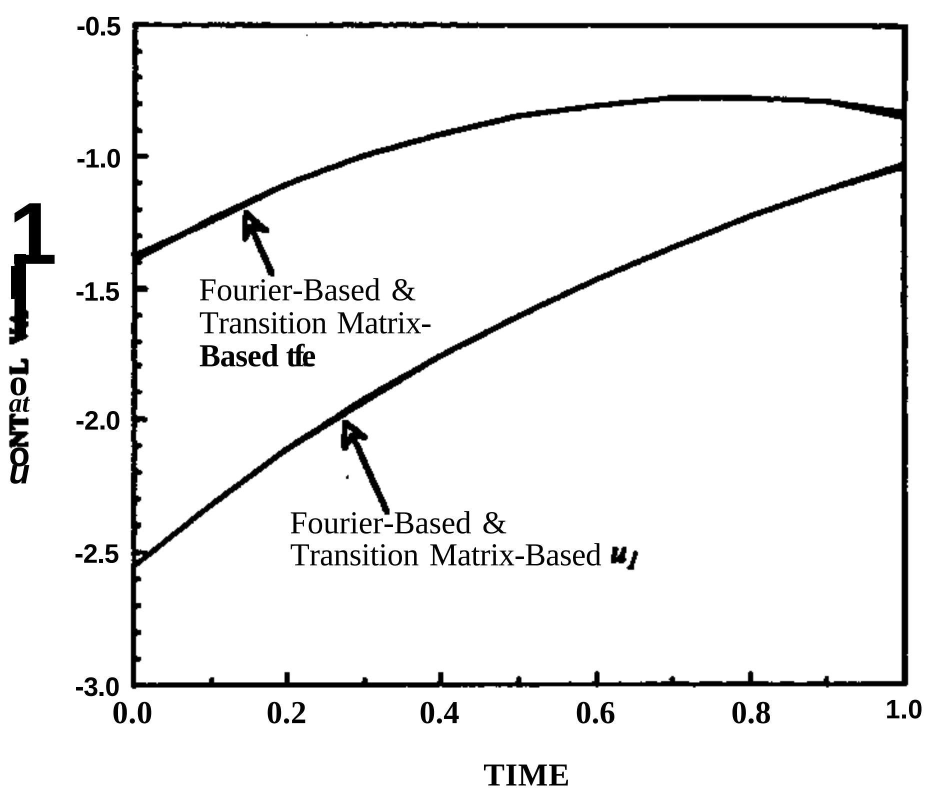

Figure lb. Control Variable Histories for Example 1

Example 2: The goal of this example is to test the Fourier-based approach for designing optimal trajectories of general linear systems. The state equation and initial condition are the same as specified in Example 1 except that the control matrix here is a column vector specified as

$$B^T = [\,0 \cdots 0 \quad 1\,] \tag{101}$$

and the performance index is

$$L = x^T(1)Hx(1) + \int_0^1 (x^T Q x + u^2)\,dt \;, \quad H = 10 I_{N \times N} \;, \quad Q = I_{N \times N} \tag{102}$$

This problem has been solved using both the transition matrix and a single segment, two-term Fourier-based approach for $N = 2, 3, ..., 16$. The weighting constant r of the artificial control was chosen to be 10^5. The simulation results, summarized in Table II, show that the Fourier-based approach is again computationally more efficient in handling high order systems. The time responses of the state and control variables for the case $N = 2$ are plotted in Figures. 2a and 2b, respectively. These figures show that the solutions from both approaches are hardly distinguishable.

Table II: Summary of Simulation Results of Example 2

	Transition-Matrix Approach		**Fourier-Based Approach"**		**Comparison**	
N	**Performance Index**	Time	**Performance Index**	**Time**	**%Time***	*A%L*[C]
2	27.358	0.22	**27.362**	0.38	**172.8**	**< 1.5xlO-**[2]
3	**195.033**	0.44	195.171	0.66	**150.0**	< 6.6x10-2
4	**705.569**	**0.87**	**706.255**	1.04	**135.1**	< 9.8 x 10-2
5	**1720.550**	1.43	**1721.381**	1.54	**107.7**	< 4.9x10-2
6	**3460.001**	**2.20**	**3462.970**	**2.31**	**105.0**	< 8.6x10-2
7	6027.753	**3.35**	**6030.865**	3.24	**96.7**	< 5.2x10-2
8	**9578.606**	**5.05**	**9587.778**	4.45	**88.1**	< 9.6 x 10-2
9	14415.109	**6.92**	**14443.140**	**5.98**	**86.4**	< 7.8 x 10-2
10	**20308.134**	**9.23**	**20331.694**	**7.80**	**84.5**	< 1.2x10-1
11	281422.031	12.74	**28176.018**	**9.89**	**77.6**	< 1.3 x 10-1
12	36881.498	**16.20**	**36933.743**	12.41	**76.6**	< 1.5 x 10-1
13	48453.432	**20.32**	**48537.693**	**15.22**	74.9	< 1.8 x 10-1
14	60525.689	**26.48**	**60628.971**	18.45	**69.7**	< 2.8 x 10-1
15	**76593.643**	**32.30**	76772.657	**22.19**	**68.7**	< 2.4x10-1
16	**92466.982**	40.48	**92653.710**	**26.70**	**66.0**	< 2.1 x 10-1

***With single segment two-term Fourier-type series**

^Percent of execution time of Fourier-based approach relative to execution time of transition-matrix approach

Percent difference of performance index of Fourier-based approach relative to performance index value of transition-matrix approach

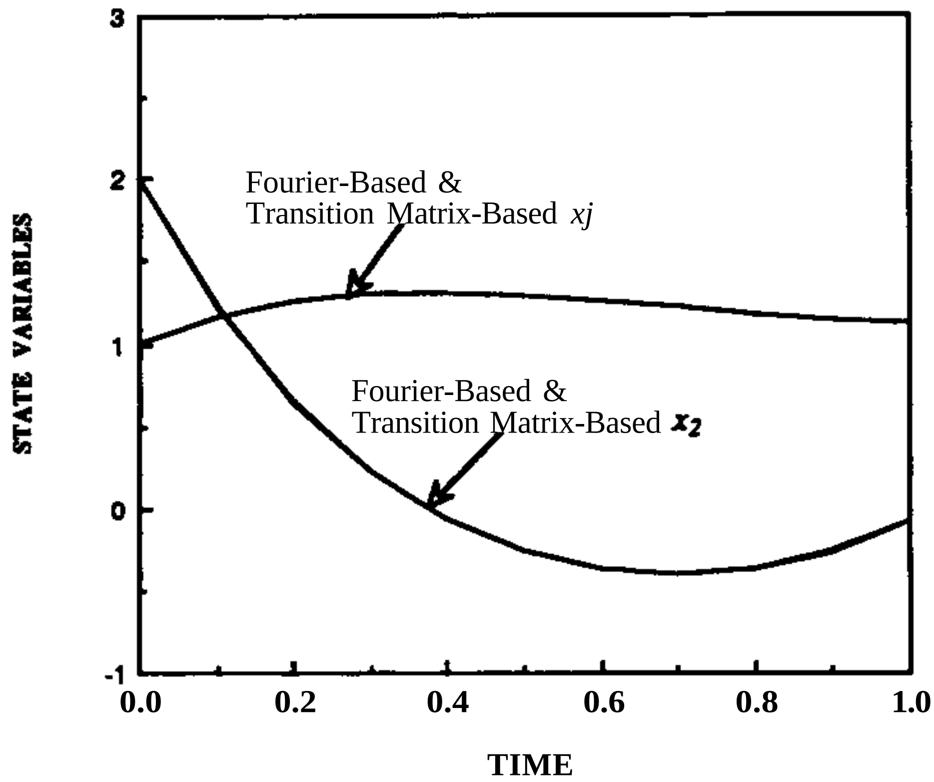

Figure 2a. State Variable Histories for Example 2

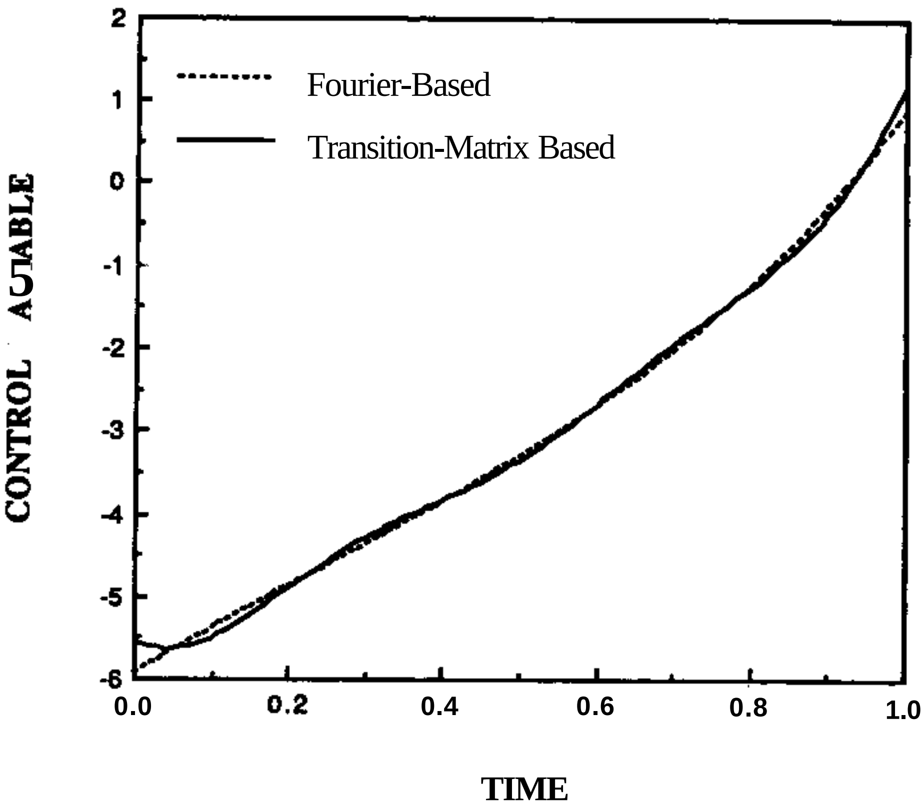

Figure 2b. Control Variable Histories for Example 2

Example 3: This example is adapted from (Evtushenko, 1985, p. 438). Here, a LQ problem with a linear state constraint is considered. The system is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{103}$$

The performance index is given as

$$L = \int_0^1 \left(x_1^2 + x_2^2 + 0.005u^2 \right) dt \tag{104}$$

The optimal solution is required to minimize this performance index without violating the following constraint

$$x_1(t) \le e(t) \tag{105}$$

where

$$e(t) = 8(t - 0.5)^2 - 0.5 \tag{106}$$

This problem was solved using a one segment Fourier-based approach. The resulting response curves for $x_1(t)$ obtained with three, five, and seven term Fourier-type series are plotted in Figure 3a, 3b and 3c, respectively. The solution computed by (Evtushenko, 1985) is also plotted in these figures for comparison. The minimum performance index obtained by (Evtushenko, 1985) is 0.17114. The performance index values determined from the Fourier-based approach are summarized in Table III. The seven term Fourier-based solution provides the best results.

Table III: Summary of Simulation Results of Example 3 using Single Segment *K* Term Fourier-Type Series (Evtushenko's Solution gives 0.171140)

K	Performance Index
3	0.174797
5	0.171154
7	0.170692

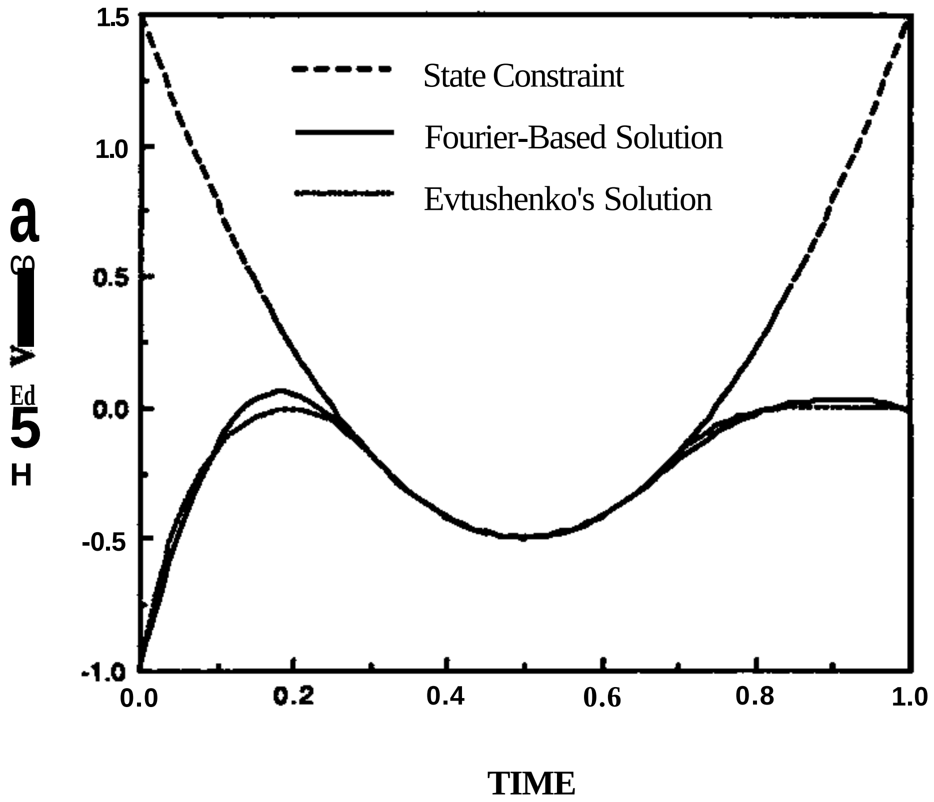

Figure 3a. State Variable x_j History for Example 3 (With Three Term Fourier-Type Series)

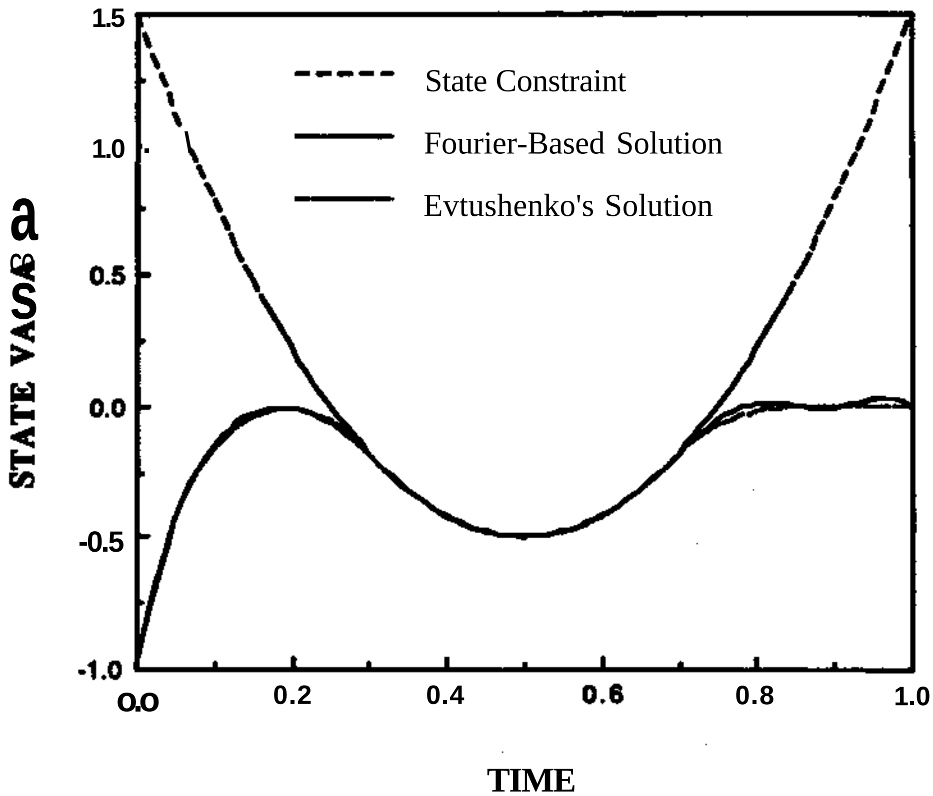

Figure 3b. State Variable *xj* History for Example 3 (With Five Term Fourier-Type Series)

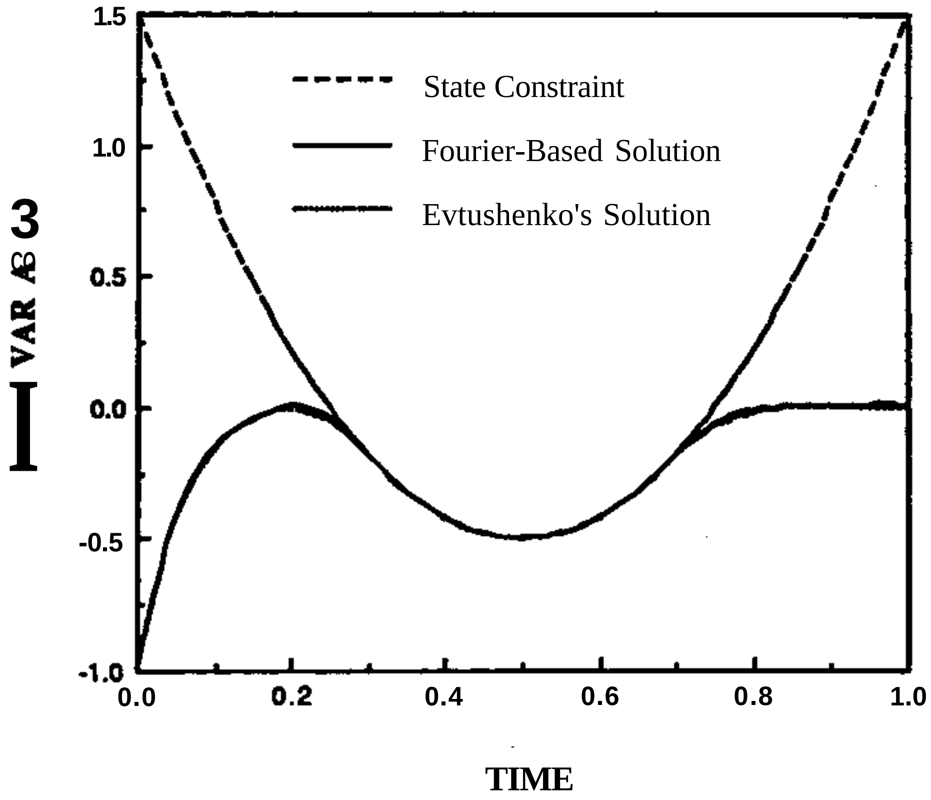

Figure 3c. State Variable *xi* History for Example 3 (With Seven Term Fourier-Type Series)

Example 4: This example is adapted from (Leondes and Wu, 1971). Here, the system is described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u,\quad \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}0.231\\ \text{[illegible]}\end{bmatrix} \tag{107}$$

The performance index is

$$L=\frac{1}{2}\int_0^5\left(x_1^2+x_2^2\right)dt \tag{108}$$

The constraint is imposed on the control variable as

$$|u|\leq 0.8 \tag{109}$$

The optimal solution, as computed by (Leondes and Wu, 1971) has a bang-bang nature, ***i.e.,***

$$u(t)=-0.8 \quad for \quad 0\leq t<1.275\,,\quad u(t)=0.8 \quad for \quad 1.275<t\leq 5.0 \qquad (110),(111)$$

The corresponding value of the performance index is 5.660.

This problem was first solved using a one segment Fourier-based approach. The control variable response histories obtained using three, six, and nine term Fourier-type series are plotted in Figure 4a. The values of the performance index obtained by a three to nine term single segment Fourier-type series are tabulated in Table IV. From this table and

Table IV: Summary of Simulation Results of Example 4 using Single Segment *K* Term Fourier-Type Series (Optimal Value is 5.660)

K	Performance **Index**
3	8.288
4	7.982
5	7.004
6	6.600
7	6.464
8	6.295
9	**6.141**

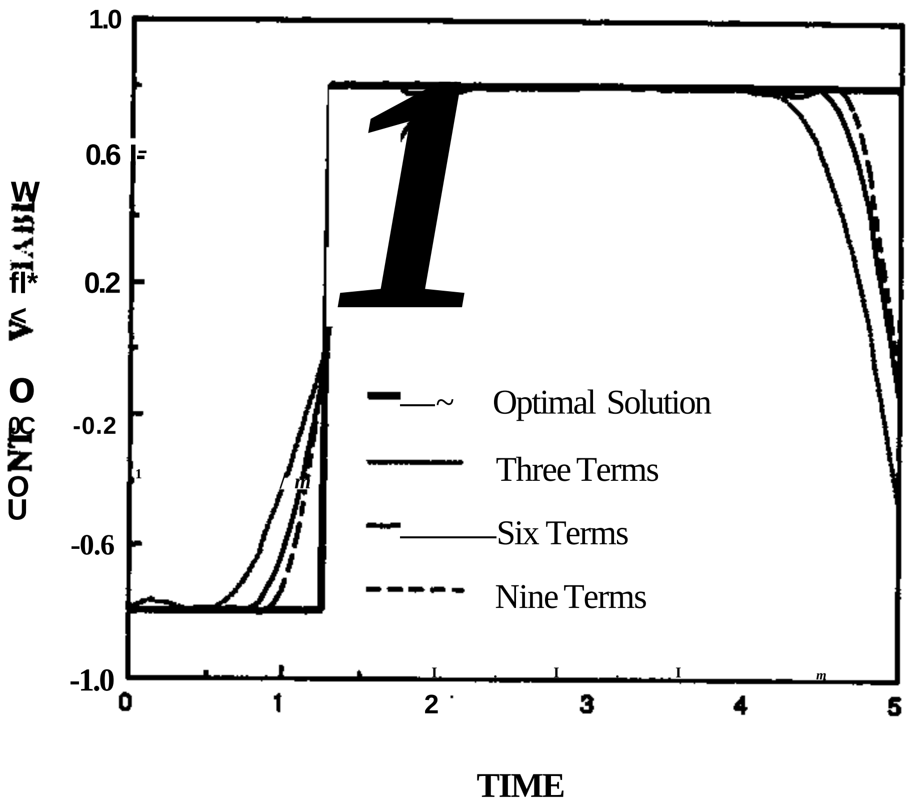

Figure 4a. Control Variable Histories for Example 4
One Segment Fourier-Based Approach

Figure 4a, it is seen that the near optimal solutions generated by the Fourier-based state parameterization approach converge to the true optimal bang-bang solution as the number of terms of the Fourier-type series increases. However, the speed of convergence is quite slow. The principal reason for this slow convergence is due the instantaneous switch of the control variable of the optimal solution at $t = 1.275$. In contrast, the Fourier-based approach assumes continuity throughout the trajectory. Consequently, significant discrepancies between the true and near optimal solution can be observed in the neighborhood of the point of the finite jump.

One remedy of this slow convergence is the application of the multiple segment Fourier-based approach. The idea here is to first estimate the locations of the instantaneous jumps by using the single segment Fourier-based approach, and then represent each continuous part of the trajectory by a unique Fourier-based representation. In this case, the time interval [0,5] is divided into two intervals [0,1.3] and [1.3,5.0] and a three term, double segment Fourier-based approach is applied. The resulting performance index value is 6.027 which is less than the single segment solutions listed in Table IV. The control variable response of the double segment solution is plotted in Figure 4b.

The quality of the Fourier-based solution can be improved further by increasing the number of segments. For instance, in this example, the time interval can be divided into three segments, [0,1.2],[1.2,1.3] and [1.3,5.0], where the point of the finite jump falls in the second segment. In particular, includes the finite jumps in a unique segment enables the close approximation of the instantaneous shift. The result is given in Figure 6 where the optimal control trajectory and the near optimal control trajectory of a three term, three segment Fourier-based approach are plotted. The performance index value of this Fourier-based solution is 5.747 which has less than a 2% error compared to the true optimal value.

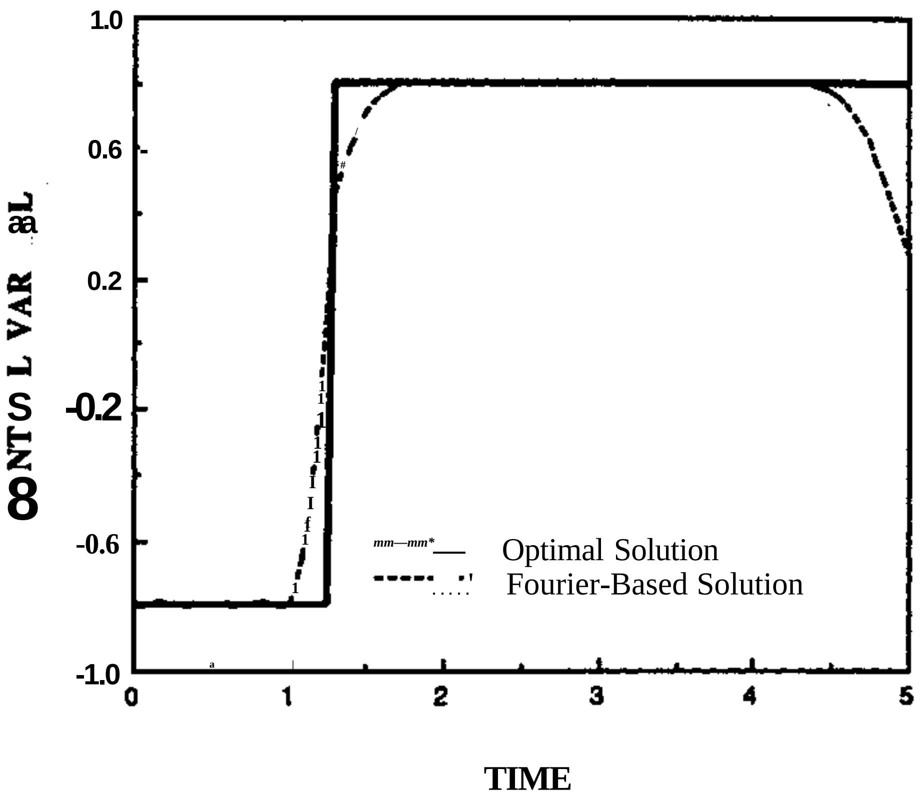

Figure 4b. Control Variable History for Example 4
Two Segment, Three Term Fourier-Based Approach

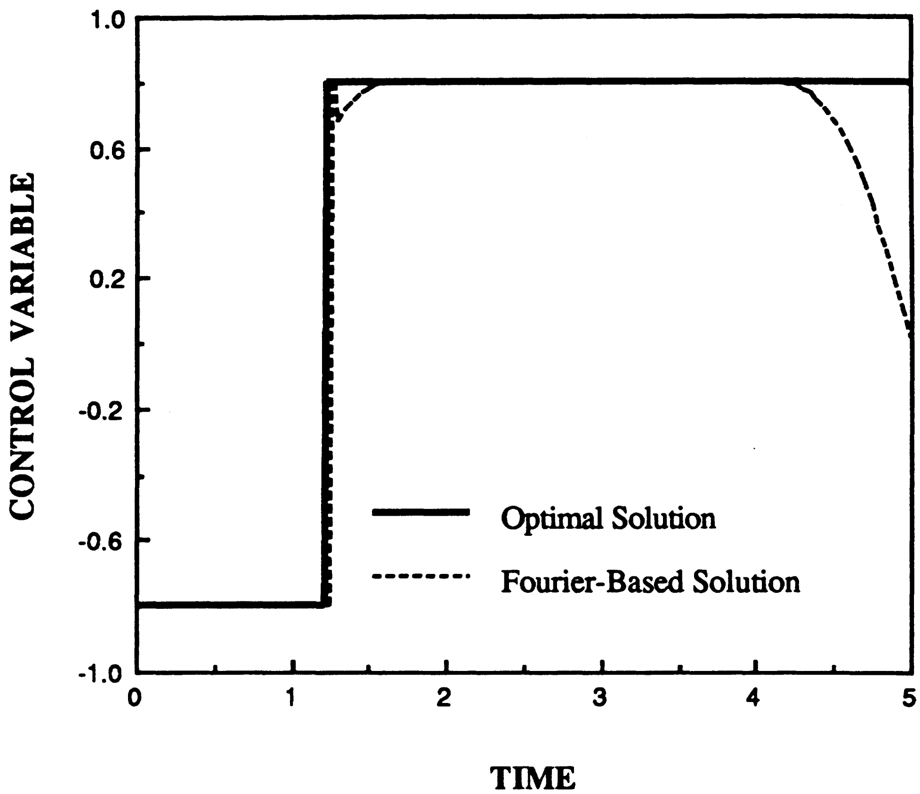

Figure 4c. Control Variable History for Example 4
Three Segment, Three Term Fourier-Based Approach

Conclusion

Based on the idea of state trajectory parameterization, this technical report develops a Fourier-based design tool for determining optimal trajectories of LQ problems. It is shown how a LQ problem can be converted to a QP problem. In particular, for an unconstrained LQ problem the necessary condition of optimality is obtained by differentiating the converted quadratic performance index with respect to free state parameters. The computational simplicity of the approach is due to the fact that the necessary condition of optimality can be derived as a system of linear algebraic equations. Simulation results indicate that the Fourier-based approach is more efficient than the standard transition matrix approach in handling high order unconstrained LQ problems.

Simulation studies also show that, in many cases, a single segment Fourier-based approximation provides sufficient accuracy when the optimal solution is continuous. A multiple segment Fourier-based approximation is required only for problems whose optimal solution has discontinuities which are not generally physically implementable. An advantage of the Fourier-based approach is that it provides an accurate and continuous near optimal solution. In summary, by relying upon well developed QP solution algorithms, the Fourier-based state parameterization approach promises to be an effective and general computational tool for designing trajectories of linearly constrained LQ systems.

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