NUMBER SEQUENCES

by

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THESIS

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A B S T R A C T

Some preliminary theorems about numbers are proved, and some particular types of number sequences are shown to have sequential limits. It is then shown that if a sequence has a sequential limit, that sequence is convergent, and that if a sequence is convergent, it has a sequential limit; two necessary and sufficient conditions for absolute convergence of a number sequence are established.
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Certain Properties of Numbers</td>
<td>6</td>
</tr>
<tr>
<td>II. Some Particular Sequences Which Have Sequential Limits</td>
<td>9</td>
</tr>
<tr>
<td>III. Convergence and Sequential Limits</td>
<td>14</td>
</tr>
<tr>
<td>IV. Absolute Convergence</td>
<td>21</td>
</tr>
</tbody>
</table>
CERTAIN PROPERTIES OF NUMBERS

The following five theorems are intended to establish certain properties of real numbers which will be useful in proving the later theorems concerning sequences.

Theorem 1. If \( p \) is a positive number, then there is an integer \( q \) such that \( q > p \).

**Proof:** Suppose \( p \) is a number such that if \( q \) is an integer, then \( q < p \).

Let \( Q \) denote the set such that \( X \) belongs to \( Q \) if and only if \( X \) is an integer. \( Q \) has upper bound \( p \), so \( Q \) has a least upper bound \( u \).

If \( e > 0 \), \( u - e < u \) so there exists a positive integer \( n \) such that \( n > u - e \); in particular \( \frac{1}{2} > 0 \) so there exists an integer \( n \) such that \( n > u - \frac{1}{2} \); so \( n + \frac{1}{2} > u \).

But \( n + 1 \) is an integer, and \( n + 1 > n + \frac{1}{2} \), so \( n + 1 > u \), i.e., \( u \) is not an upper bound of \( Q \). Therefore, the supposition that \( Q \) has an upper bound \( p \) must be false.

Theorem 2. If \( p > 0 \), then there is a positive integer \( n \) such that \( 1/n < p \).

**Proof:** Suppose there is a number \( p \) such that if \( n \) is a positive integer, \( 1/n > p \). Then if \( n \) is a positive integer, \( 1/p > n \); but ac-
According to Theorem 1, there is an integer \( m \) such that \( m > 1/p \). Therefore, the supposition that there is such a number as \( p \) must be false.

**Theorem 3.** Suppose that \( p \) is a non-negative real number such that \( q > 0 \), then \( p \leq q \). Then \( p = 0 \).

**Proof:** Either \( p = 0 \), or \( p > 0 \).

If \( p > 0 \), then \( p \) satisfies the requirement on \( q \) in the hypothesis of this theorem, so \( p < p \). This cannot be true, so \( p = 0 \).

**Theorem 4.** If \( p > 0 \) and \( n \) is a positive integer, then \((1 + p)^n \geq 1 + np\).

**Proof:**

\[
(1 + p)^1 = 1 + 1 \cdot p
\]

\[
(1 + p)^2 = 1 + 2p + p^2 \quad \text{and} \quad p^2 > 0,
\]

so

\[
(1 + p)^2 > 1 + 2p
\]

\[
(1 + p)^3 = (1 + p) (1 + p)^2 > (1 + p) (1 + 2p) = 1 + 3p + 2p^2
\]

so

\[
(1 + p)^3 > 1 + 3p
\]

and, continuing this process, for each positive integer \( n \) if

\[
(1 + p)^n \geq 1 + (n - 1)p, \quad (1 + p) > 0, \quad \text{so}
\]

\[
(1 + p)^n \geq (1 + p) [1 + (n - 1)p] = 1 + np + (n - 1)p^2
\]

and \((n - 1)p^2 \geq 0,\)

so

\[
(1 + p)^n \geq 1 + np
\]

**Theorem 5.** If \( 0 < r < 1 \) and \( p > 0 \), then there is a positive number \( N \) such that \( 0 < r^N < p \) if \( n \) is an integer greater than \( N \).

**Proof:** \( r < 1 \), so \( 1 < 1/r \), i.e., there is a positive number \( k \) such that \( 1/r = 1 + k \). Therefore, if \( n \) is a positive integer, \((1/r)^n = 1/r^n = (1 + k)^n\),
and, by Theorem 4, $(1 + k)^n \geq 1 + nk$.

$\frac{1}{kp} > 0$, so, by Theorem 1, there is a positive integer $n$ such that $n > \frac{1}{kp}$. If $n > \frac{1}{kp}$, $nk > \frac{1}{p}$, so $1 + nk > 1 + \frac{1}{p}$. Therefore, $1/r^n > 1 + \frac{1}{p}$.

$1 + \frac{1}{p} > \frac{1}{p}$, so $1/r^n > \frac{1}{p}$, or $p > r^n$. Moreover, for each positive integer $n$, $r^n$ is the product of $n$ positive numbers, so $r^n > 0$. Therefore, if $N$ denotes the number $\frac{1}{kp}$ and $n$ is an integer greater than $N$, $0 < r^n < p$. 
CHAPTER II

SOME PARTICULAR SEQUENCES WHICH HAVE SEQUENTIAL LIMITS

Definition 1. Suppose that $S_1, S_2, S_3, \ldots$ is a number-sequence. The statement that $S$ is the sequential limit of $S_1, S_2, S_3, \ldots$ means that $S$ is a number and that if $p>0$, then there is a positive number $N$ such that $|S_n - S| < p$ if $n$ is an integer greater than $N$.

Lemma 6a. If $r$ is a number, and $r \neq 1$, and $n$ is a positive integer, then

$$\frac{1}{1-r} = 1 + r + r^2 + \ldots + r^{n-1} + r^n \frac{1}{1-r}.$$  

Proof: 

$$\frac{1}{1-r} = (1-r + r^2 + \ldots + r^n \frac{1}{1-r})$$  

$$= 1 + r \frac{1}{1-r}$$  

$$= 1 + (r - r^2 + r^3) \frac{1}{1-r}$$  

$$= 1 + r (r - r^2 + r^3) \frac{1}{1-r}$$  

and, continuing this process, $r^{n-1} \frac{1}{1-r}$.

$$= \frac{r^{n-1} - r^n \frac{1}{1-r}}{1-r}$$  

$$= \frac{r^{n-1}}{1-r} (1-r) + r^n \frac{1}{1-r}$$  

$$= \frac{r^{n-1} + r^n \frac{1}{1-r}}{1}$$ for each positive integer $n$, so

$$\frac{1}{1-r} = 1 + r + r^2 + \ldots + r^{n-1} + r^n \frac{1}{1-r}.$$

Theorem 6. If $|r| < 1$ and $p>0$, then there is a positive number $N$ such that $|1 + r + r^2 + \ldots + r^{n-1} - 1/\frac{1}{1-r}| < p$, if $n$ is an integer greater than $N$.

Proof: By Lemma 6a, $1 + r + r^2 + \ldots + r^{n-1} + r^n \frac{1}{1-r} = \frac{1}{1-r}$.
so, \((1 + r + r^2 + \ldots + r^{n-1}) - 1/\left(1-r\right) = r^n/\left(1-r\right)\).

Therefore, \(\left|\left(1 + r + r^2 + \ldots + r^{n-1}\right) - 1/\left(1-r\right)\right| = |r^n/\left(1-r\right)|.

0 < |r| < 1 and \(\frac{1}{1-r} \geq p > 0\), so, by Theorem 5, there exists a positive number \(N\) such that \(0 < r^n < \left|1-r\right|p\) if \(n\) is an integer greater than \(N\).

\[
\left|\frac{r^n}{1-r}\right| = \frac{|r^n|}{|1-r|} = \frac{|r|^n}{1-r} \quad \text{so if } n \text{ is an integer greater than } N, \quad |r|^n < |1-r|p
\]

and \(\left|\frac{r^n}{1-r}\right| = \frac{|r^n|}{|1-r|} < |1-r|p = p\). Therefore, \(\left(1 + r + r^2 + \ldots + r^{n-1}\right) - 1/\left(1-r\right)\) is the sequential limit of the sequence \(1, 1 + r, 1 + r + r^2, 1 + r^2 + r^3, \ldots\).

**Theorem 7.** Suppose that \(S_1, S_2, S_3, \ldots\) is a non-increasing positive-number sequence (i.e., \(0 < S_{n+1} \leq S_n\) for each positive integer \(n\)). There is a non-negative real number \(S\) which is the sequential limit of \(S_1, S_2, S_3, \ldots\).

**Proof:** \(S_1, S_2, S_3, \ldots\) has lower bound zero, so there is a greatest number \(S\) which exceeds no element of \(S_1, S_2, S_3, \ldots\). If \(p > 0\), there exists a positive integer \(N\) such that \(S_N < S + p\); otherwise, \(S + p\) would be a lower bound of \(S_1, S_2, S_3, \ldots\), which exceeds \(S\), the greatest lower bound of \(S_1, S_2, S_3, \ldots\).

If \(n > N\), \(S_N \geq S > S\geq S_n\geq S > 0\). But \(p > S_N - S\) so \(p > S_n - S > 0\), and \(S_n - S < p\) for each integer \(n\) greater than \(N\). Therefore, \(S\) is a sequential limit of \(S_1, S_2, S_3, \ldots\).

**Corollary 7a.** Suppose that \(S_1, S_2, S_3, \ldots\) is a bounded non-decreasing number sequence (i.e., that there is a positive number \(B\) such that \(S_n \leq S_{n+1} \leq B\) for each positive integer \(n\)). There is a real number \(S\) which is the sequential limit of \(S_1, S_2, S_3, \ldots\).
Proof: Let $B'$ denote $B + 1$. Then, $B' > B$, so if $S_n \leq S_{n+1} \leq B$, $S_n \leq S_{n+1} < B'$. For each positive integer $n$, $S_n \leq S_{n+1} < B'$, so $-S_n \geq -S_{n+1} > -B'$ and $B - S_n \geq B - S_{n+1} > 0$, so $B' - S_1, B' - S_2, B' - S_3, \ldots$ is a non-increasing positive number sequence. Therefore, by Theorem 7, there is a non-negative real number $k$ which is the sequential limit of $B' - S_1, B' - S_2, B' - S_3, \ldots$.

If $p > 0$, there is a positive number $N$ such that if $n$ is an integer greater than $N, |B' - S_n - k| < p$. Let $S$ denote $B - k$. Then $-k = S - B$ and $|B' - S_n - k| = |B' - S_n + S - B| = |S_n - S|$. Therefore, if $n > N, |S_n - S| < p$; that is, $S$ is the sequential limit of $S_1, S_2, S_3, \ldots$.

**Theorem 8.** Suppose that $S_1, S_2, S_3, \ldots$ is a non-increasing number sequence and that $S_2, S_3, S_4, \ldots$ is a non-decreasing number sequence. If $S_1, S_2, S_3, \ldots$ has a sequential limit $S$, then $S_{2n-1} \geq S \geq S_{2n}$ for each positive integer $n$. If $S_{2n-1} \geq S_{2n}$ for each positive integer $n$, then for $S_1, S_2, S_3, \ldots$ to have a sequential limit, it is necessary and sufficient that the following statement be true: if $P > 0$, then there is a positive integer $k$ such that $|S_k - S_{k+1}| < p$.

**Proof:** Suppose $T$ is a number and, for some positive integer $n$, $T \geq S_{2n-1}$. $S_1, S_3, \ldots, S_{2n-1}, \ldots$ is a non-increasing number sequence so if $k$ is an integer greater than $n$, $S_{2n-1} \geq S_{2k-1}$. If $p = T - S_{2n-1}$, $P > 0$, and if $N$ is a positive number there is an integer $k$ such that $k > 1/2 \left( N + 2n \right)$, (i.e., $2k - 1 > N + 2n - 1$). If $k$ is such an integer, $2k - 1 > N$, but $|S_{2k-1} - T| > T - S_{2n-1}$. Therefore, for each positive number $N$ there is an integer $2k - 1$ greater than $N$ such that $|S_{2k-1} - T| > p$. So if $S$ is a sequential limit of $S_1, S_2, S_3, \ldots$, it must be true that $S_{2n+1} \geq S$ for each positive integer $N$. A similar argument shows that if $S$ is a sequential limit of $S_1, S_2, S_3, \ldots$, $S \geq S_{2n}$ for each positive integer $n$. 

Suppose that \( S_1, S_2, S_3, \ldots \) has a sequential limit \( S \), and \( p > 0 \). If \( k > 0 \), so there is a positive number \( N \) such that if \( k \) is an integer greater than \( N \), \( |S_k - S| < p/2 \). If \( k > N \), \( k+1 > N \) so \( |S_{k+1} - S| < p/2 \). If \( k > N \), \( k+1 > N \) so \( |S_k - S_{k+1}| = |(S_k - S) - (S_{k+1} - S)| \leq |S_k - S| + |S_{k+1} - S| < p/2 + p/2 \), so \( |S_k - S_{k+1}| < p \) if \( k > N \).

Suppose if \( p > 0 \) there is a positive integer \( k \) such that \( |S_k - S_{k+1}| < p \) and \( S_{2n+1} \geq S_{2n} \) for each positive integer \( n \).

Suppose \( S_2, S_4, S_6, \ldots \) is a non-decreasing real number sequence, and for each \( n \), \( S_{2n+1} \geq S_{2n} \).

\( S_1 \geq S_{2n+1} \) since \( S_1, S_3, S_5, \ldots \) is non-increasing. Therefore, \( S_{2n} \leq S_{2n+2} \leq S_1 \) and so, by Corollary 7a, \( S_2, S_4, S_6, \ldots \) has a sequential limit \( S \).

There is a positive number \( N' \) such that \( |S_{2n} - S| < p/2 \), if \( n \) is an integer greater than \( N'/2 \). By hypothesis there is a positive integer \( k' \) such that \( |S_{k'} - S_{k'+1}| < p/2 \).

Suppose \( m \) is an integer greater than \( k' + N' \). If \( k' \) is odd and \( m \) is odd, \( S_k \geq S_m \) since \( S_1, S_3, S_5, \ldots \) is non-increasing and \( S_{k'+1} \leq S_{m+1} \) since \( S, S_2, S_4, S_6, \ldots \) is non-decreasing. From the hypothesis of the theorem we know that \( S_m \geq S_{m+1} \), so \( |S_m - S_{m+1}| = S_m - S_{m+1} \leq S_{k'} - S_{k'+1} = |S_{k'} - S_{k'+1}| \leq p/2 \). \( |S_m - S| \leq |S_m - S_{m+1}| + |S_{m+1} - S| \) and \( m+1 \) is an even integer greater than \( N' \), so \( |S_{m+1} - S| < p/2 \) and \( |S_m - S| < p/2 + p/2 \), or \( |S_m - S| < p \).
If $k'$ is even and $m$ is odd, $S_{k'} \geq S_{m+1}$ because $S, S_3, S_5, \ldots$ is non-increasing, and $S_{k'+1} \leq S_m$ because $S_2, S_4, S_6, \ldots$ is non-decreasing.

$S_m \geq S_{m+1}$, so $|S_m - S_{m+1}| = S_m - S_{m+1} \leq S_{k'+1} - S_{k'} = |S_{k'} - S_{k'+1}| \leq \frac{p}{2}$.

$|S_m - S| \leq |S_m - S_{m+1}| + |S_{m+1} - S|$ and $m+1$ is an even integer greater than $n$ so $|S_{m+1} - S| \leq \frac{p}{2}$, and $|S_m - S| \leq \frac{p}{2} + \frac{p}{2}$, so $|S_m - S| \leq p$.

Therefore, if $p > 0$ and $m$ is an integer greater than $N' + k'$, $|S_m - S| \leq p$, so $S$ is a sequential limit of $S_1, S_2, S_3, \ldots$. 
CHAPTER III

CONVERGENCE AND SEQUENTIAL LIMITS

Definition 2. Suppose that \( S_1, S_2, S_3, \ldots \) is a number sequence. The statement that \( S_1, S_2, S_3, \ldots \) converges means that if \( p \) is a positive number then there is a positive number \( N \) such that \( |S_m - S_n| < p \) if \( m \) and \( n \) are integers greater than \( N \).

Theorem 9. If the sequence \( S_1, S_2, S_3, \ldots \) has a sequential limit, then the sequence converges.

Proof: By hypothesis, \( S_1, S_2, S_3, \ldots \) has a sequential limit \( S \), i.e., if \( p > 0 \) there exists a positive number \( N \) such that \( |S_n - S| < p \) if \( n \) is an integer greater than \( N \).

If \( p > 0 \), \( p/2 > 0 \), so there exists a positive number \( N^* \) such that if \( n \) is an integer greater than \( N^* \), \( |S_n - S| < p/2 \). Suppose \( m \) and \( n \) are integers greater than \( N^* \). Then, \( |S_m - S_n| \leq |S_m - S| + |S_n - S| < p/2 \) and \( |S_n - S| < p/2 \), so \( |S_m - S_n| < p \), if \( m \) and \( n \) are integers greater than \( N^* \).

Definition 3. The statement that \( K \) is a disk with center \( c \) and radius \( r \) means that \( c \) is a number, \( r \) is a positive number, and \( K \) is a number set such that the number \( Z \) is in \( K \) if and only if \( |Z - c| \leq r \).

Theorem 10. Suppose that \( K_1 \) is the disc with center \( c_1 \) and radius \( r_1 \). For \( K_2 \) to be a subset of \( K_1 \) (that is to say, for each member of \( K_2 \) to be a member of \( K_1 \)), it is necessary and sufficient that \( |c_1 - c_2| \leq r_1 - r_2 \).
Proof: Suppose \(|c_1 - c_2| \leq r_1 - r_2\) and \(Z\) is in \(K_2\), i.e., \(|Z - c_2| \leq r_2\). Then \(|Z - c_1| = |Z - c_2 + c_2 - c_1|\), but \(|Z - c_2 + c_2 - c_1| \leq |Z - c_2| + |c_2 - c_1|\), so \(|Z - c_1| \leq r_2 + r_1 - r_2\), \(|Z - c_1| \leq r_1\), i.e., \(K_2\) is a subset of \(K_1\).

Suppose \(K_2\) is a subset of \(K_1\), i.e., if \(|Z - c_2| \leq r_2\), then \(|Z - c_1| \leq r_1\). Then \(|(c_2 + r_2 \frac{c_2 - c_1}{|c_2 - c_1|}) - c_2| = r_2\), so \(|(c_2 + r_2 \frac{c_2 - c_1}{|c_2 - c_1|}) - c_1| = |(c_2 - c_1) + (c_2 - c_1) \frac{r_2}{|c_2 - c_1|}| = |c_2 - c_1| \cdot |1 + \frac{r_2}{|c_2 - c_1|}| = \frac{|c_2 - c_1| \cdot |c_2 - c_1| + r_2}{|c_2 - c_1|} = |c_2 - c_1| + r_2\) \(\leq r_1\). Therefore, since \((|c_2 - c_1| + r_2) > 0\), \(|c_2 - c_1| + r_2\) = \(|c_2 - c_1| + r_2\), so \(|c_2 - c_1| + r_2 \leq r_1\), or, \(|c_2 - c_1| \leq r_1 - r_2\).

**Definition 4.** The statement that \(K_1, K_2, K_3, \ldots\) is a nest of discs means that if \(n\) is a positive integer, then \(K_n\) is a disc and \(K_{n+1}\) is a subset of \(K_n\).
Theorem 11. Suppose that $K_1, K_2, K_3, \ldots$ is a nest of disks. There is a number $c$ such that if $n$ is a positive integer, $c$ is in $K_n$.

Proof: For each positive $n$, let $r_n$ denote the radius and $c_n$ denote the center of $K_n$. By Theorem 10, $|c_r - c_{n+1}| \leq r_n - r_{n+1}$ for each positive integer $n$, so $r_1, r_2, r_3, \ldots$ is a non-increasing positive number sequence. Therefore, by Theorem 7, $r_1, r_2, r_3, \ldots$ has a non-negative sequential limit $r$.

Suppose $r > 0$. Then $r/2 > 0$, so there exists a positive number $N_1$ such that if $n$ is an integer greater than $N_1$, $|r_n - r| < r/2$. Further, by Theorem 9, $r_1, r_2, r_3, \ldots$ converges, so there exists a positive number $N_2$ such that if $m$ and $n$ are integers greater than $N_2$, $|r_m - r_n| < r/2$. Let $N$ denote $N_1 + N_2$. Then if $m > n > N$, $K_m$ is a subset of $K_n$, so, by Theorem 10, $|c_n - c_m| \leq r_n - r_m < r/2$. But $|r_m - r| < r/2$, or $r - r/2 \leq r_m < r + r/2$, so $r/2 \leq r_m$. Therefore, if $n^*$ denotes a particular integer greater than $N$, then $|c_n - c_m| < r_n$ for each positive integer $m$ greater than $n^*$; that is, $c_n$ belongs to $K_m$ for each positive integer $m$ greater than $n^*$. But $c_n$ belongs to $K_n$, and $K_n$ is a subset of each element of the sequence $K_1, K_2, K_3, \ldots$ which precedes it, so for each positive integer $n$, $c_n$ belongs to $K_n$.

Suppose $r = 0$. If $Z$ is a number, there is a real number $x$ and there is a real number $y$ such that $Z = x + iy$. It is also true that $|Z| \geq |x|$ and $|Z| \geq |y|$. For each positive integer $n$, let $a_n$ denote a real number and $b_n$ denote a real number such that $c_n = a_n + b_n i$. 
For each positive integer $n$, $K_{n+1}$ is a subset of $K_n$, so $|c_n - c_{n+1}| \leq r_n - r_{n+1}$.

$r_n - r_{n+1} \cdot |c_n - c_{n+1}| = |(a_n - a_{n+1}) + (b_n - b_{n+1})|$, so $|a_n - a_{n+1}| \leq |c_n - c_{n+1}| r_n - r_{n+1}$, or $a_n - (r_n - r_{n+1}) \leq a_{n+1} + (r_n - r_{n+1})$.

Therefore, $a_n - a_{n+1} \leq r_{n+1} - r_n$, and $a_{n+1} + r_{n+1} \leq a_n + r_n$.

Let $S_{2n-1}$ denote $a_n + r_n$ and $S_{2n}$ denote $a_n - r_n$. Then if $p > 0$, $p/2 > 0$, and the number sequence $r_1$, $r_2$, $r_3$, ... has sequential limit 0, so there exists a positive number $N$, such that if $k$ is an integer greater than $N$, $|r_k - 0| = |r_k| < p/2$. Suppose $k$ is such an integer. Then $|S_{2k} - S_{2k-1}| = |(a_k - r_k) - (a_k + r_k)| = |2r_k| < p$. For each positive integer $n$, $r_n > 0$, so $S_{2k-1} = a_k + r_k \geq a_k - r_k = S_{2k}$. Therefore, the hypothesis of Theorem 8 is satisfied, and $S_1$, $S_2$, $S_3$, ... has a sequential limit $a$.

There exists a positive number $N_2$ such that if $n$ is an integer greater than $N_2$, $|S_n - a| < p/2$. Suppose $N = 2(N_1 + N_2)$ and $m'$ is an integer greater than $N$. Then there is an integer $m$ greater than $N_1 + N_2$ such that either $m' = 2m$ or $m' = 2m-1$.

If $m' = 2m$, $|S_m - a| = |a_m - r_m - a| \geq |a_m - a| - |r_m|$, so $|a_m - a| \leq |S_m - a| + |r_m| < p/2 + p/2 = p$. If $m' = 2m-1$, $|S_m - a| = |a_m + r_m - a| \geq |a_m - a| - |r_m|$, so $|a_m - a| \leq |S_m - a| + |r_m| < p$. Therefore, $a$ is a sequential limit of $a_1$, $a_2$, $a_3$, ... . This same argument can be used to show that $b_1$, $b_2$, $b_3$, ... has a sequential limit $b$. 
Suppose \( c = a + bi \). Then if \( p > 0 \), there is a positive number \( N_1 \) such that if \( n \) is an integer greater than \( N_1 \), \( |a_n - a| < p/2 \), and there is a positive number \( N_2 \) such that if \( n \) is an integer greater than \( N_2 \), \( |b_n - b| < p/2 \), so if \( n \) is an integer greater than \( N_1 + N_2 \), \( |c_n - c| = |(a_n - a) + (b_n - b)i| \leq |a_n - a| + |b_n - b| < p/2 + p/2 = p \), so \( c \) is a sequential limit of \( c_1, c_2, c_3, \ldots \).

Suppose \( c_m \) is the center of some disk \( K_n \) of the nest of disks \( K_1, K_2, K_3, \ldots \) and that \( c \) does not belong to \( K_n \). Then \( |c_n - c| = r_n + q \), where \( q > 0 \), \( |c_n - c_n| + |c_m - c| \geq |c_n - c| \), so \( |c_n - c_n| + |c_m - c| \geq r_n + q \). Therefore, if \( c \) did not belong to some disk of the nest \( K_1, K_2, K_3, \ldots \) \( c \) would not be the sequential limit of \( c_1, c_2, c_3, \ldots \).

**Theorem 12.** If \( S_1, S_2, S_3, \ldots \) is a convergent number sequence, there exists a nest \( K_1, K_2, K_3, \ldots \) of disks such that:

1. If \( n \) is a positive integer, \( S_n \) is in \( K_n \), and
2. If \( p > 0 \), there exists a positive number \( N \) such that if \( n \) is an integer greater than \( N \) and \( r_n \) is the radius of \( K_n \), \( r_n < p \).

**Proof:** For each positive integer \( \frac{1}{k}, 1/k > 0 \), so there exists a positive integer \( N \) such that if \( m \) and \( n \) are integers greater than \( N \), \( |S_m - S_n| < 1/k \). For each positive integer \( k \), let \( N_k \) denote one such number. Then let \( n_1 \)
denote an integer greater than $N_1$ and for each positive integer $k$ greater than one, let $n_k$ denote an integer greater than $N_k + n_{k-1}$.

If $n$ is an integer and $n < n_{k+1}$, let $K_n$ denote the disk with center $S_n$ and radius $r_n = 1/k\left[1 + 1/n\right]$.

Then, if for some integer $k$, $n$ and $n + 1$ are both greater than $n_k$ and less than or equal to $n_{k+1}$, 
\[
1/k\left[1 + 1/n\right] - 1/k\left[1 + 1/n + 1\right] = 1/k\left[n + 2\right] - \frac{n + 2}{n + 1} = 1/k\left[1/n(\frac{1}{n+1})\right] = 1/k\left[1/(n+1)\right] \geq \left|S_{n_k} - S_{n+1}\right|,
\]
so $K_{n+1}$ is a subset of $K_n$.

If for some integer $k$, $n = n_k$, then 
\[
1/k\left[1 + 1/n\right] - 1/k\left[1 + 1/n + 1\right] = 1/k\left[-1/n\right] - 1/k\left[1 + 1/n + 1\right] = 1/k\left[1/n(\frac{1}{n+1})\right] = 1/k\left[1/(n+1)\right] \geq \left|S_{n_k} - S_{n+1}\right|,
\]
so $K_{n+1}$ is a subset of $K_n$.

Therefore, $K_1, K_2, K_3, ...$ is a nest of disks.

**Theorem 13.** Suppose that the number sequence $S_2, S_3, S_4, ...$ converges. There is a number $S$ which is the sequential limit of $S_2, S_3, S_4, ...$

**Proof:** By Theorem 12, there is a nest of disks $K_1, K_2, K_3, ...$ such that if $n$ is a positive integer, $S_n \in K_n$, and such that if $p > 0$, there is a positive number $N$ such that if $n$ is an integer greater than $N$ and $r_n$ is the radius of $K_n$, then $r_n < p$.

By Theorem 11, there is a number $S$ such that if $n$ is a positive integer, $S$ is in $K_n$. 

Therefore, if \( p > 0 \), and \( N \) denotes a positive number such that if \( n \) is an integer greater than \( N \), \( r_n \leq p \), then \( |S_n - S| < p \). Therefore, \( S \) is a sequential limit of \( S_1, S_2, S_3, \ldots \).
CHAPTER IV

ABSOLUTE CONVERGENCE

Definition 5. Suppose that $C_1$, $C_2$, $C_3$, ... is a number sequence. The statement that $C_1$, $C_2$, $C_3$, ... converges absolutely means that if $S_1 = |C_2 - C_1|$ and $S_{n+1} = S_n + |C_{n+1} - C_n|$ for each positive integer $n$, then the sequence $S_1, S_2, S_3, ...$ converges.

Theorem 14. For the number sequence $C_1$, $C_2$, $C_3$, ... to converge absolutely, it is necessary and sufficient that there be a nest, $K_1, K_2, K_3, ...$ of disks such that $C_n$ is the center of $K_n$, $n = 1, 2, 3, ...$

Proof: Suppose the sequence $C_1$, $C_2$, $C_3$, ... converges absolutely, i.e., that if $S_1 = |C_2 - C_1|$ and $S_{n+1} = S_n + |C_{n+1} - C_n|$ for each positive integer $n$, then the sequence $S_1, S_2, S_3, ...$ converges.

By Theorem 13, $S_1, S_2, S_3, ...$ has a sequential limit $S$. $S_1, S_2, S_3, ...$ is non-decreasing so $S$ must be not less than $S_n$ for each positive integer $n$.

Therefore, $1 + S - S_n$ is a positive number for each positive integer $n$, so for each positive integer $n$, let $K_n$ denote the disk with center $C_n$ and radius $1 + S - S_n$. $|C_{n+2} - C_{n+1}| = S_{n+1} - S_n = (1 + S - S_n) - (1 + S - S_{n+1})$, so by Theorem 10, $K_{n+2}$ is a subset of $K_{n+1}$. If $K_1$ denotes the disk with center $C_1$ and radius $1 + S$, $|C_2 - C_1| = S_1 = (1 + S) - (1 + S - S_1)$, so by Theorem 10, $K_2$ is a subset of $K_1$. Therefore, $K_1, K_2, K_3, ...$ is a nest of disks having centers $C_1, C_2, C_3, ...$.
Suppose there is a nest $K_1, K_2, K_3, \ldots$ of disks such that $C_n$ is the center of $K_n$ for each positive integer $n$. $S_1 = |C_2 - C_1| \leq r_1 - r_2$

$S_2 = S_1 + |C_3 - C_2| \leq (r_1 - r_2) + (r_2 - r_3) = r_1 - r_3$, $S_3 = S_2 + |C_4 - C_3| \leq (r_1 - r_3) + (r_3 - r_4) = r_1 - r_4$, \ldots

$S_{n+1} = S_n + |C_{n+2} - C_{n+1}| \leq (r_1 - r_{n+1}) + (r_{n+1} - r_{n+2}) = r_1 - r_{n+2}$

$r_1, r_2, r_3, \ldots$ is a non-increasing positive number sequence so by Theorem 7 there is a non-negative real number $r$ such that $r$ is the sequential limit of $r_1, r_2, r_3, \ldots$. For each positive integer $n$, $r_n \leq r$, so $r_1 - r_n \leq r_1 - r$.

$S_{n+1} = S_n + |C_{n+2} - C_{n+1}| \leq r_1 - r_{n+2} \leq r_1 - r$, so $S_n \leq S_{n+1} \leq r_1 - r$, and, by Corollary 7a, there is a real number $S$ which is the sequential limit of $S_1, S_2, S_3, \ldots$.

**Theorem 15.** Suppose that $C_1, C_2, C_3, \ldots$ is a number sequence such that $C_{n+1} \neq C_n$, $n = 1, 2, 3, \ldots$. For $C_1, C_2, C_3, \ldots$ to converge absolutely, it is necessary and sufficient that there be a positive number sequence $d_1, d_2, d_3, \ldots$ such that

$$\frac{|C_{n+2} - C_{n+1}|}{|C_{n+1} - C_n|} \leq \frac{d_n}{1 + d_{n+1}}, \quad n = 1, 2, 3, \ldots$$

Moreover, if $d_1, d_2, d_3, \ldots$ is such a sequence and $C$ is the sequential limit of $C_1, C_2, C_3, \ldots$ then

$$|C_{n+1} - C| \leq d_n |C_{n+1} - C_n|, \quad n = 1, 2, 3, \ldots$$

**Proof:** Suppose $S_1 = |C_2 - C_1|$ and $S_{n+1} = S_n + |C_{n+2} - C_{n+1}|$, $n = 1, 2, 3, \ldots$

Suppose there is a positive number sequence $d_1, d_2, d_3, \ldots$ such that
Then, if \( n = 1 \), \( \frac{d_1}{1 + d_2} > \frac{S_2 - S_1}{S_1} \), and if \( n \) is an integer greater than 1, \( \frac{d_n}{1 + d_{n+1}} > \frac{S_{n+1} - S_n}{S_n - S_{n-1}} \).

\( S_2, S_3, \ldots \) is an increasing number sequence, i.e., \( S_{n+1} - S_n > 0 \), for each positive integer \( n \), \( d_1 S \geq (1 + d_2) (S_2 - S_1) = S_2 - S_1 + d_2 (S_2 - S_1) \),

\( d_2 (S_2 - S_1) \geq (1 + d_3) (S_3 - S_2) = S_3 - S_2 + d_3 (S_3 - S_2) \),

so \( d_1 S \geq S_3 - S_1 + d_3 (S_3 - S_2) \), \( d_3 (S_3 - S_2) \geq (1 + d_4) (S_4 - S_3) = S_4 - S_3 + d_4 (S_4 - S_3) \), so \( d_1 S \geq S_4 - S_1 + d_4 (S_4 - S_3) \), and if \( d_1 S \geq S_{n-1} S - S_1 + d_{n-1} (S_{n-1} - S_n) \), \( d_{n-1} (S_{n-1} - S_n) \geq (1 + d_n) (S_n - S_{n-1}) = S_n - S_{n-1} + d_n (S_n - S_{n-1}) \), so \( d_1 S \geq S_n - S_1 + d_n (S_n - S_{n-1}) \), and \( d_n (S_n - S_{n-1}) > 0 \). Therefore, \((1 + d_1) S \geq S_n\) for each positive integer \( n \), and so by Corollary 7a \( S_1, S_2, S_3, \ldots \) has a sequential limit \( S \). By Theorem 9, the sequence \( S_1, S_2, S_3, \ldots \) converges, so \( C_1, C_2, C_3, \ldots \) is absolutely convergent.

Suppose that the sequence \( C_1, C_2, C_3, \ldots \) is absolutely convergent, i.e., that the sequence \( S_1, S_2, S_3, \ldots \) is convergent. By Theorem 13, \( S_1, S_2, S_3, \ldots \) has a sequential limit \( S \), and \( S_1, S_2, S_3, \ldots \) is increasing so \( S > S_n \) for each positive integer \( n \).

If for each integer \( n \) greater than 1, \( d_n = \frac{S - S_n}{S_n - S_{n-1}} \), then \( d_n > 0 \) and

\[
\frac{d_{n+1}}{S_{n+1} - S_n} = \frac{S - S_{n+1}}{S_n - S_{n-1}} + 1 + d_{n+1} = \frac{S_{n+1} - S_n}{S_n} + \frac{S - S_{n+1}}{S_{n+1} - S_n} = \frac{S - S_n}{S_n - S_{n-1}}.
\]
Therefore, \( d_n = \frac{S - S_n}{1 + d_{n+1}} \cdot \frac{S_{n+1} - S_n}{S - S_n} = \frac{S_{n+1} - S_n}{S - S_{n-1}} = \frac{|C_{n+2} - C_{n+1}|}{|C_{n+1} - C_n|} \)

If \( d_1 = S - S_1 \) then \( d_1 > 0 \), and \( 1 + d_2 = S - S_1 \) so \( d_1 = \frac{S - S_1}{1 + d_2} \)

\[
\frac{S - S_1}{S} \cdot \frac{S_2 - S_1}{S - S_1} = \frac{S - S_1}{S_1} = \frac{|C_3 - C_2|}{|C_2 - C_1|}.
\]

Therefore, there is a positive number sequence \( d_1, d_2, d_3, \ldots \) such that

\[
\frac{d_n}{1 + d_{n+1}} \geq \frac{|C_{n+2} - C_{n+1}|}{|C_{n+1} - C_n|}, \quad n = 1, 2, 3, \ldots
\]

Suppose \( C \) denotes the sequential limit of \( C_1, C_2, C_3, \ldots \).

If \( n \) is a positive integer, \( d_n \left| C_{n+1} - C_n \right| \geq (1 + d_n) \left| C_{n+2} - C_{n+1} \right| = \left| C_{n+2} - C_{n+1} \right| + d_n \left| C_{n+2} - C_{n+1} \right|\)

\[
d_n \left| C_{n+1} - C_n \right| \geq d_n \left| C_{n+2} - C_{n+1} \right| + d_n \left| C_{n+3} - C_{n+2} \right| = d_n \left| C_{n+3} - C_{n+2} \right| + d_n \left| C_{n+4} - C_{n+3} \right|.
\]

So, \( d_n \left| C_{n+1} - C_n \right| \geq d_n \left| C_{n+2} - C_{n+1} \right| + d_n \left| C_{n+3} - C_{n+2} \right| + d_n \left| C_{n+4} - C_{n+3} \right| \)

but \( \left| C_{n+2} - C_{n+1} \right| + \left| C_{n+3} - C_{n+2} \right| \geq \left| C_{n+3} - C_{n+1} \right| \), so \( d_n \left| C_{n+1} - C_n \right| \geq \left| C_{n+3} - C_{n+1} \right| \)

\[
d_n \left| C_{n+1} - C_n \right| \geq \left| C_{n+3} - C_{n+2} \right|.
\]

\[
d_n \left| C_{n+3} - C_{n+2} \right| \geq (1 + d_{n+3}) \left| C_{n+4} - C_{n+3} \right| = \left| C_{n+4} - C_{n+3} \right| + d_{n+3} \left| C_{n+4} - C_{n+3} \right| + d_{n+3} \left| C_{n+5} - C_{n+4} \right|.
\]

By continuing this process, we find that if \( k \) is a positive integer, \( d_n \left| C_{n+1} - C_n \right| \geq \left| C_{n+1+k} - C_{n+1} \right| + d_{n+k} \left| C_{n+1+k} - C_{n+k} \right| \).
Let \( q \) denote the least non-negative number such that \( d_n \left| C_{n+1} - C_n \right| + q \geq \left| C_{n+1} - C \right| \). Then if \( k \) is a positive integer, \( \left| C_{n+1+k} - C \right| \geq q \).

\( C \) is the sequential limit of \( C_1, C_2, C_3, \ldots \) so if \( p > 0 \), there exists a positive number \( N \) such that if \( m \) is an integer greater than \( N \), \( p > \left| C_m - C \right| \).

If \( m \) is greater than \( N \), \( n+1+m > N \), so \( p > \left| C_{n+1+m} - C \right| \geq q \), i.e., if \( p > 0 \), \( p > q \). Therefore, by Theorem 3, \( q = 0 \) and \( d_n \left| C_{n+1} - C_n \right| \geq \left| C_{n+1} - C \right| \).
VITA

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