NUMBER SEQUENCES

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THESIS

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A B S T R A C T

Some preliminary theorems about numbers are proved, and some particular types of number sequences are shown to have sequential limits. It is then shown that if a sequence has a sequential limit, that sequence is convergent, and that if a sequence is convergent, it has a sequential limit; two necessary and sufficient conditions for absolute convergence of a number sequence are established.

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CHAPTER I

CERTAIN PROPERTIES OF NUMBERS

The following five theorems are intended to establish certain properties of real numbers which will be useful in proving the later theorems concerning sequences.

__Theorem 1.__ If p is a positive number, then there is an integer q such that $q > p$.

__Proof:__ Suppose p is a number such that if q is an integer, then $q < p$.

Let Q denote the set such that X belongs to Q if and only if X is an integer. Q has upper bound p, so Q has a least upper bound u.

If $e > o$, $u - e < u$ so there exists a positive integer n such that $n > u - e$; in particular $\frac{1}{2} > o$ so there exists an integer n such that $n > u - \frac{1}{2}$; so $n + \frac{1}{2} > u$.

But $n + 1$ is an integer, and $n + 1 > n + \frac{1}{2}$, so $n + 1 > u$, i.e., u is not an upper bound of Q. Therefore, the supposition that Q has an upper bound p must be false.

__Theorem 2.__ If $p > o$, then there is a positive integer n such that $1/n < p$.

__Proof:__ Suppose there is a number p such that if n is a positive integer, $1/n > p$. Then if n is a positive integer, $1/p > n$; but according to Theorem 1, there is an integer m such that $m > 1/p$. Therefore, the supposition that there is such a number as p must be false.

Theorem 3. Suppose that p is a non-negative real number such that if $q > o$, then $p < q$. Then $p = o$.

Proof: Either $p = o$, or $p > o$.

If $p > o$, then p satisfies the requirement on q in the hypothesis of this theorem, so $p < p$. This cannot be true, so $p = o$.

Theorem 4. If $p > o$ and n is a positive integer, then $(1 + p)^n \geq 1 + np$.

Proof: $(1 + p)^1 = 1 + 1 \cdot p$

$$(1 + p)^2 = 1 + 2p + p^2 \quad \text{and } p^2 > o,$$

so $$(1 + p)^2 > 1 + 2p$$

$$(1 + p)^3 = (1 + p)\,(1 + p)^2 > (1 + p)\,(1 + 2p) = 1 + 3p + 2p^2$$

so $$(1 + p)^3 > 1 + 3p$$

and, continuing this process, for each positive integer n if $(1 + p)^{n - 1} \geq 1 + (n - 1)p$, $(1 + p) > o$, so

$$(1 + p)^n \geq (1 + p)\,[1 + (n - 1)p] = 1 + np + (n - 1)p^2$$

and $(n - 1)p^2 \geq o$,

so $$(1 + p)^n \geq 1 + np$$

Theorem 5. If $O < r < 1$ and $p > O$, then there is a positive number N such that $O < r^n < p$ if n is an integer greater than N.

Proof: $r < 1$, so $1 < 1/r$, i.e., there is a positive number k such that $1/r = 1 + k$. Therefore, if n is a positive integer, $(1/r)^n = 1/r^n = (1 + k)^n$, and, by <u>Theorem 4</u>, $(1 + k)^n \geq 1 + nk$.

$1/kp > 0$, so, by Theorem 1, there is a positive integer n such that $n > 1/kp$. If $n > 1/kp$, $nk > 1/p$, so $1 + nk > 1 + 1/p$. Therefore, $1/r^n > 1 + 1/p$. $1 + 1/p > 1/p$, so $1/r^n > 1/p$, or $p > r^n$. Moreover, for each positive integer n, r^n is the product of n positive numbers, so $r^n > 0$. Therefore, if N denotes the number $1/kp$ and n is an integer greater than N, $0 < r^n < p$.

CHAPTER II

SOME PARTICULAR SEQUENCES WHICH HAVE SEQUENTIAL LIMITS

Definition 1. Suppose that $S_1, S_2, S_3, \ldots$ is a number-sequence. The statement that S is the sequential limit of $S_1, S_2, S_3, \ldots$ means that S is a number and that if $p > 0$, then there is a positive number N such that $|S_n - S| < p$ if n is an integer greater than N.

Lemma 6a. If r is a number, and $r \neq 1$, and n is a positive integer, then $1/(1-r) = 1 + r + r^2 + r^3 + \ldots + r^{n-1} + r^n/(1-r)$.

Proof: $1/(1-r) = (1-r + r)/(1-r)$

$= 1 + r/(1-r)$

$= 1 + (r - r^2 + r^2)/(1-r)$

$= 1 + r + r^2/(1-r)$, and, continuing this process, $r^{n-1}/(1-r)$

$= (r^{n-1} - r^n + r^n)/(1-r)$

$= (r^{n-1}(1-r) + r^n)/(1-r)$

$= r^{n-1} + r^n/(1-r)$ for each positive integer n, so

$$1/(1-r) = 1 + r + r^2 + r^3 + \ldots + r^{n-1} + r^n/(1-r)$$

Theorem 6. If $|r| < 1$ and $p > 0$, then there is a positive number N such that $|(1 + r + r^2 + \ldots + r^{n-1}) - 1/(1-r)| < p$ if n is an integer greater than N.

Proof: By Lemma 6a, $1 + r + r^2 + \ldots + r^{n-1} + r^n/(1-r) = 1/(1-r)$

so, $(1 + r + r^2 + \dots + r^{n-1}) - 1/(1-r) = r^n/(1-r)$.

Therefore, $|(1 + r + r^2 + \dots + r^{n-1}) - 1/(1-r)| = |r^n/(1-r)|$.

$0 < |r| < 1$ and $|1-r|\, p > 0$, so, by Theorem 5, there exists a positive number N such that $0 < |r|^n < |1-r|\, p$ if n is an integer greater than N.

$\left|\frac{r^n}{1-r}\right| = \frac{|r^n|}{|1-r|} = \frac{|r|^n}{|1-r|}$, so if n is an integer greater than N, $|r|^n < |1-r|\, p$ and $\left|\frac{r^n}{1-r}\right| = \frac{|r|^n}{|1-r|} < \frac{|1-r|\, p}{|1-r|} = p$. Therefore, $|(1 + r + r^2 + \dots + r^{n-1}) - 1/(1-r)| < p$, or $1/(1-r)$ is the sequential limit of the sequence 1, $1 + r$, $1 + r + r^2$, $1 + r + r^2 + r^3$, ...

Theorem 7. Suppose that $S_1, S_2, S_3, \dots$ is a non-increasing positive-number sequence (i.e., $0 < S_{n+1} \leq S_n$ for each positive integer n). There is a non-negative real number S which is the sequential limit of $S_1, S_2, S_3, \dots$.

Proof: $S_1, S_2, S_3, \dots$ has lower bound zero, so there is a greatest number S which exceeds no element of $S_1, S_2, S_3, \dots$. If $p > 0$, there exists a positive integer N such that $S_N < S + p$; otherwise, $S + P$ would be a lower bound of $S_1, S_2, S_3, \dots$ which exceeds S, the greatest lower bound of $S_1, S_2, S_3, \dots$.

If $n > N$, $S_N \geq S_n > S$ and $S \geq 0$, so $S_N - S \geq S_n - S > 0$. But $p > S_N - S$ so $p > S_n - S > 0$, and $|S_n - S| < p$ for each integer n greater than N. Therefore, S is a sequential limit of $S_1, S_2, S_3, \dots$.

Corollary 7a. Suppose that $S_1, S_2, S_3, \dots$ is a bounded non-decreasing number sequence (i.e., that there is a positive number B such that $S_n \leq S_{n+1} \leq B$ for each positive integer n). There is a real number S which is the sequential limit of $S_1, S_2, S_3, \dots$.

Proof: Let B' denote $B+1$. Then, $B'>B$, so if $S_n \leq S_{n+1} \leq B$, $S_n \leq S_{n+1} < B'$.

For each positive integer n, $S_n \leq S_{n+1} < B'$, so $-S_n \geq -S_{n+1} > -B'$ and $B'-S_n \geq B'-S_{n+1} > 0$, so $B' - S_1$, $B' - S_2$, $B' - S_3$, ... is a non-increasing positive number sequence. Therefore, by Theorem 7, there is a non-negative real number k which is the sequential limit of $B' - S_1$, $B' - S_2$, $B' - S_3$, ...

If $p > 0$, there is a positive number N such that if n is an integer greater than N, $|B' - S_n - k| < p$. Let S denote B - k. Then $-k = S - B$ so $|B' - S_n - k| = |B' - S_n + S - B'| = |S_n - S|$. Therefore, if $n > N$, $|S_n - S| < p$; that is, S is the sequential limit of S_1, S_2, S_3, ...

<u>Theorem 8</u>. Suppose that S_1, S_3, S_5,... is a non-increasing number sequence and that S_2, S_4, S_6,... is a non-decreasing number sequence. If S_1, S_2, S_3,... has a sequential limit S, then $S_{2n-1} \geq S \geq S_{2n}$ for each positive integer n. If $S_{2n-1} \geq S_{2n}$ for each positive integer n, then for S_1, S_2, S_3,... to have a sequential limit it is necessary and sufficient that the following statement be true: if $P>0$, then there is a positive integer k such that $|S_k - S_{k+1}| < p$.

<u>Proof:</u> Suppose T is a number and, for some positive integer n, $T > S_{2n-1}$. S_1, S_3, ... , S_{2n-1}, ... is a non-increasing number sequence so if k is an integer greater than n, $S_{2n-1} \geq S_{2k-1}$. If $p = T-S_{2n-1}$, $p>0$, and if N is a positive number there is an integer k such that $k > 1/2\ (N + 2n)$, (i.e., $2k - 1 > N + 2n - 1$). If k is such an integer, $2k - 1 > N$, but $|S_{2k-1} - T| > T - S_{2n-1}$. Therefore, for each positive number N there is an integer 2k-1 greater than N such that $|S_{2k-1} - T| > p$. So if S is a sequential limit of S_1, S_2, S_3,... it must be true that $S_{2n+1} \geq S$ for each positive integer N. A similar argument shows that if S is a sequential limit of S_1, S_2, S_3,..., $S \geq S_{2n}$ for each positive integer n.

Suppose that $S_1, S_2, S_3, \ldots$ has a sequential limit S, and $p > 0$. $p/2 > 0$, so there is a positive number N such that if k is an integer greater than N, $|S_k - S| < p/2$. If $k > N$, $k+1 > N$ so $|S_{k+1} - S| < p/2$. $|S_k - S_{k+1}| = |(S_k - S) - (S_{k+1} - S)| \leq |S_k - S| + |S_{k+1} - S| < p/2 + p/2$, so $|S_k - S_{k+1}| < p$ if $k > N$.

Suppose if $p > 0$ there is a positive integer k such that $|S_k - S_{k+1}| < p$ and $S_{2n+1} \geq S_{2n}$ for each positive integer n.

$S_2, S_4, S_6, \cdots$ is a non-decreasing real number sequence, and for each n, $S_{2n+1} \geq S_{2n}$.

$S_1 \geq S_{2n+1}$ since $S_1, S_3, S_5, \ldots$ is non-increasing. Therefore, $S_{2n} \leq S_{2n+2} \leq S_1$ and so, by Corollary 7a, $S_2, S_4, S_6, \ldots$ has a sequential limit S.

There is a positive number N' such that $|S_{2n} - S| < p/2$, if n is an integer greater than $N'/2$. By hypothesis there is a positive integer k' such that $|S_{k'} - S_{k'+1}| < p/2$.

Suppose m is an integer greater than $k' + N'$. If k' is odd and m is odd, $S_{k'} \geq S_m$ since $S_1, S_3, S_5, \ldots$ is non-increasing and $S_{k'+1} \leq S_{m+1}$ since $S, S_2, S_4, S_6, \ldots$ is non-decreasing. From the hypothesis of the theorem we know that $S_m \geq S_{m+1}$, so $|S_m - S_{m+1}| = S_m - S_{m+1} \leq S_{k'} - S_{k'+1} = |S_{k'} - S_{k'+1}| < p/2$. $|S_m - S| \leq |S_m - S_{m+1}| + |S_{m+1} - S|$ and $m+1$ is an even integer greater than N' so $|S_{m+1} - S| < p/2$ and $|S_m - S| < p/2 + p/2$, or $|S_m - S| < p$.

If k' is even and m is odd, $S_{k'} \geq S_{m+1}$ because S_1, S_3, S_5, . . . is non-increasing, and $S_{k'+1} \leq S_m$ because S_2, S_4, S_6,... is non-decreasing. $S_m \geq S_{m+1}$, so $|S_m - S_{m+1}| = S_m - S_{m+1} \leq S_{k'+1} - S_{k'} = |S_{k'} - S_{k'+1}| < p/2$. $|S_m - S| \leq |S_m - S_{m+1}| + |S_{m+1} - S|$ and $m+1$ is an even integer greater than n so $|S_{m+1} - S| < p/2$, and $|S_m - S| < p/2 + p/2$, so $|S_m - S| < p$.

Therefore, if $p > 0$ and m is an integer greater than $N' + k'$, $|S_m - S| < p$, so S is a sequential limit of S_1, S_2, S_3, ...

CHAPTER III

CONVERGENCE AND SEQUENTIAL LIMITS

Definition 2. Suppose that $S_1, S_2, S_3, \ldots$ is a number sequence. The statement that $S_1, S_2, S_3, \ldots$ converges means that if p is a positive number then there is a positive number N such that $|S_m - S_n| < p$ if m and n are integers greater than N.

Theorem 9. If the sequence $S_1, S_2, S_3, \ldots$ has a sequential limit, then the sequence converges.

Proof: By hypothesis, $S_1, S_2, S_3, \ldots$ has a sequential limit S, i.e., if $p > 0$ there exists a positive number N such that $|S_n - S| < p$ if n is an integer greater than N.

If $p > 0$, $p/2 > 0$, so there exists a positive number N' such that if n is an integer greater than N', $|S_n - S| < p/2$. Suppose m and n are integers greater than N'. Then, $|S_m - S_n| \leq |S_m - S| + |S_n - S|$. $|S_m - S| < p/2$ and $|S_n - S| < p/2$, so $|S_m - S_n| < p$, if m and n are integers greater than N'.

Definition 3. The statement that K is a disk with center c and radius r means that c is a number, r is a positive number, and K is a number set such that the number Z is in K if and only if $|Z - c| \leq r$.

Theorem 10. Suppose that K_1 is the disc with center c_1 and radius r_1. For K_2 to be a subset of K_1 (that is to say, for each member of K_2 to be a member of K_1,) it is necessary and sufficient that $|c_1 - c_2| \leq r_1 - r_2$.

<u>Proof</u>: Suppose $|C_1 - C_2| \leq r_1 - r_2$ and Z is in K_2, i.e., $|Z - C_2| \leq r_2$. Then $|Z - C_1| = |Z - C_2 + C_2 - C_1|$, but $|Z - C_2 + C_2 - C_1| \leq |Z - C_2| + |C_2 - C_1|$, and $|Z - C_2| \leq r_2$ and $|C_2 - C_1| = |C_1 - C_2| \leq r_1 - r_2$, so $|Z - C_1| \leq r_2 + r_1 - r_2$. $|Z - C_1| \leq r_1$, i. e., K_2 is a subset of K.

Suppose K_2 is a subset of K_1, i.e., if $|Z - C_2| \leq r_2$, then $|Z - C_1| \leq r_1$. Then $\left|\left(C_2 + r_2 \frac{C_2 - C_1}{|C_2 - C_1|}\right) - C_2\right| = r_2$, so $\left|\left(C_2 + r_2 \frac{C_2 - C_1}{|C_2 - C_1|}\right) - C_1\right| \leq r_1$, but $\left|\left(C_2 + r_2 \frac{C_2 - C_1}{|C_2 - C_1|}\right) - C_1\right| = \left|(C_2 - C_1) + (C_2 - C_1)\frac{r_2}{|C_2 - C_1|}\right|$ $= |C_2 - C_1| \cdot \left|1 + \frac{r_2}{|C_2 - C_1|}\right| = \frac{|C_2 - C_1|}{|C_2 - C_1|} \cdot \left||C_2 - C_1| + r_2\right| = \left||C_2 - C_1| + r_2\right| \leq r_1$. Therefore, since $(|C_2 - C_1| + r_2) > 0$, $\left||C_2 - C_1| + r_2\right| = |C_2 - C_1| + r_2$, so $|C_2 - C_1| + r_2 \leq r_1$, or, $|C_2 - C_1| \leq r_1 - r_2$.

<u>Definition 4</u>. The statement that K_1, K_2, K_3, ... is a nest of discs means that if n is a positive integer, then K_n is a disc and K_{n+1} is a subset of K_n.

Theorem 11. Suppose that K_1, K_2, K_3, ... is a nest of disks. There is a number c such that if n is a positive integer, c is in K_n.

Proof: For each positive n, let r_n denote the radius and c_n denote the center of K_n. By Theorem 10, $|c_r - c_{n+1}| \leq r_n - r_{n+1}$ for each positive integer n, so r_1, r_2, r_3, ... is a non-increasing positive number sequence. Therefore, by Theorem 7, r_1, r_2, r_3, ... has a non-negative sequential limit r.

Suppose $r > 0$. Then $r/2 > 0$, so there exists a positive number N_1 such that if n is an integer greater than N_1, $|r_n - r| < r/2$. Further, by Theorem 9, r_1, r_2, r_3, ... converges, so there exists a positive number N_2 such that if m and n are integers greater than N_2, $|r_m - r_n| < r/2$. Let N denote $N_1 + N_2$. Then if $m > n > N$, K_m is a subset of K_n, so, by Theorem 10, $|c_n - c_m| \leq r_n - r_m < r/2$. But $|r_m - r| < r/2$, or $r - r/2 \leq r_m \leq r + r/2$, so $r/2 \leq r_m$. Therefore, if n' denotes a particular integer greater than N, then $|c_{n'} - c_m| \leq r_m$ for each positive integer m greater than n'; that is, $c_{n'}$ belongs to K_m for each positive integer m greater than n'. But $c_{n'}$ belongs to $K_{n'}$ and $K_{n'}$ is a subset of each element of the sequence K_1, K_2, K_3, ... which precedes it, so for each positive integer n, $c_{n'}$ belongs to K_n.

Suppose $r = 0$. If Z is a number, there is a real number x and there is a real number y such that $Z = x + iy$. It is also true that $|Z| \geq |x|$ and $|Z| \geq |y|$. For each positive integer n, let a_n denote a real number and b_n denote a real number such that $c_n = a_n + b_n i$.

For each positive integer n, K_{n+1} is a subset of K_n, so $|c_n - c_{n+1}| \leq r_n - r_{n+1}$. $|c_n - c_{n+1}| = |(a_n - a_{n+1}) + (b_n - b_{n+1})i|$, so $|a_n - a_{n+1}| \leq |c_n - c_{n+1}| \leq r_n - r_{n+1}$, or $a_n - (r_n - r_{n+1}) \leq a_{n+1} \leq a_n + (r_n - r_{n+1})$. Therefore, $a_n - r_n \leq a_{n+1} - r_{n+1}$, and $a_{n+1} + r_{n+1} \leq a_n + r_n$.

Let S_{2n-1} denote $a_n + r_n$ and S_{2n} denote $a_n - r_n$. Then if $p > 0$, $p/2 > 0$, and the number sequence $r_1, r_2, r_3, \ldots$ has sequential limit 0, so there exists a positive number N, such that if k is an integer greater than N, $|r_k - 0| = |r_k| < p/2$. Suppose k is such an integer. Then $|S_{2k} - S_{2k-1}| = |(a_k - r_k) - (a_k + r_k)| = |-2r_k| < p$. For each positive integer n, $r_n > 0$, so $S_{2k-1} = a_k + r_k \geq a_k - r_k = S_{2k}$. Therefore, the hypothesis of Theorem 8 is satisfied, and $S_1, S_2, S_3, \ldots$ has a sequential limit a.

There exists a positive number N_2 such that if n is an integer greater than N_2, $|S_n - a| < p/2$. Suppose $N = 2(N_1 + N_2)$ and m' is an integer greater than N. Then there is an integer m greater than $N_1 + N_2$ such that either $m' = 2m$ or $m' = 2m-1$.

If $m' = 2m$, $|S_{m'} - a| = |a_m - r_m - a| \geq |a_m - a| - |r_m|$, so $|a_m - a| \leq |S_{m'} - a| + |r_m| < p/2 + p/2 = p$. If $m' = 2m-1$, $|S_{m'} - a| = |a_m + r_m - a| \geq |a_m - a| - |r_m|$, so $|a_m - a| \leq |S_{m'} - a| + |r_m| < p$. Therefore, a is a sequential limit of $a_1, a_2, a_3, \ldots$. This same argument can be used to show that $b_1, b_2, b_3, \ldots$ has a sequential limit b.

Suppose $c = a + bi$. Then if $p > 0$, there is a positive number N_1 such that if n is an integer greater than N_1, $|a_n - a| < p/2$, and there is a positive number N_2 such that if n is an integer greater than N_2, $|b_n - b| < p/2$, so if n is an integer greater than $N_1 + N_2$, $|c_n - c| = |(a_n - a) + (b_n - b)i| \leq |a_n - a| + |b_n - b| < p/2 + p/2 = p$, so c is a sequential limit of c_1, c_2, 3_3, ...

Suppose c_n is the center of some disk K_n of the nest of disks K_1, K_2, K_3, ... and that c does not belong to K_n. Then $|c_n - c| = r_n + q$, where $q > 0$. $|c_n - c_m| + |c_m - c| \geq |c_n - c|$, so $|c_n - c_m| + |c_m - c| \geq r_n + q$.

If m is an integer greater than n, K_m is a subset of K_n so $|c_n - c_m| \leq r_n$, and $r_m + |c_m - c| \geq r_n + q$, so $|c_m - c| \geq q$. Therefore, if c did not belong to some disk of the nest K_1, K_2, K_3, ... c would not be the sequential limit of c_1, c_2, c_3, ...

<u>Theorem 12</u>. If S_1, S_2, S_3, ... is a convergent number sequence, there exists a nest K_1, K_2, K_3, ... of disks such that:

1. If n is a positive integer, S_n is in K_n, and

2. If $p > 0$, there exists a positive number N such that if n is an integer greater than N and r_n is the radius of K_n, $r_n < p$.

<u>Proof</u>: For each positive integer k, $1/k > 0$, so there exists a positive integer N such that if m and n are integers greater than N, $|S_m - S_n| < 1/$.. For each positive integer k, let N_k denote one such number. Then let n_1

denote an integer greater than N_1 and for each positive integer k greater than one, let n_k denote an integer greater than $N_k + n_{k-1}$.

If n is an integer and $n_k < n \leq n_{k+1}$, let K_n denote the disk with center S_{n_k} and radius $r_n = 1/k\left[1 + 1/n\right]$.

Then, if for some integer k, n and n + 1 are both greater than n_k and less than or equal to n_{k+1}, $1/k\left[1 + 1/n\right] - 1/k\left[1 + 1/n+1\right] = 1/k\left[\frac{n+1}{n} - \frac{n+2}{n+1}\right] = \frac{1}{k}\left[\frac{n^2 + 2n + 1 - n^2 - 2n}{n(n+1)}\right] = 1/k\left[1/n(n+1)\right] > \left|S_{n_k} - S_{n_k}\right|$, so K_{n+1} is a subset of K_n.

If for some integer k, $n = n_k$, then $1/(k-1)\left[(1 + 1/n)\right] - 1/k\left[(1 + 1/n+1)\right] = 1/k-1\left[n+1/n\right] - 1/k\left[(n+2)/(n+1)\right]$, then $\left|S_{n_k} - S_{n+1}\right| < 1/(k-1)$ and $r_n - r_{n+1} = 1/(k-1)\left[1 + 1/n\right] - 1/k\left[1 + 1/(n+1)\right] = 1/(k-1)\left[(n+1)/n\right] - 1/k\left[(n+2)/(n+1)\right]$

$= \frac{k(n+1)^2 - n(n+2)(k-1)}{(k-1)\,nk} = \frac{1}{k-1}\,\frac{n^2 + 2n + k}{nk} = \frac{1}{k-1}\left(\frac{n}{k} + \frac{2}{k} + \frac{1}{nk}\right)$.

$n \geq k$, so $r_n - r_{n+1} \geq 1/(k-1) > \left|S_n - S_{n+1}\right|$, so K_{n+1} is a subset of K_n.

Therefore, $K_1, K_2, K_3, \ldots$ is a nest of disks.

<u>Theorem 13.</u> Suppose that the number sequence $S_1, S_2, S_3, \ldots$ converges. There is a number S which is the sequential limit of $S_1, S_2, S_3, \ldots$.

<u>Proof:</u> By Theorem 12, there is a nest of disks $K_1, K_2, K_3, \ldots$ such that if n is a positive integer, S_n is in K_n, and such that if $p > 0$, there is a positive number N such that if n is an integer greater than N and r_n is the radius of K_n, then $r_n < p$.

By Theorem 11, there is a number S such that if n is a positive integer, S is in K_n.

Therefore, if $p > 0$, and N denotes a positive number such that if n is an integer greater than N, $r_n < p$, then $|S_n - S| < p$. Therefore, S is a sequential limit of S_1, S_2, S_3, ...

CHAPTER IV

ABSOLUTE CONVERGENCE

Definition 5. Suppose that $C_1, C_2, C_3, \ldots$ is a number sequence. The statement that $C_1, C_2, C_3, \ldots$ converges absolutely means that if $S_1 = |C_2 - C_1|$ and $S_{n+1} = S_n + |C_{n+1} - C_{n+1}|$ for each positive integer n, then the sequence $S_1, S_2, S_3, \ldots$ converges.

Theorem 14. For the number sequence $C_1, C_2, C_3, \ldots$ to converge absolutely, it is necessary and sufficient that there be a nest, K_1, $K_2, K_3, \ldots$ of disks such that C_n is the center of K_n, $n = 1,2,3,\ldots$.

Proof: Suppose the sequence $C_1, C_2, C_3, \ldots$ converges absolutely, i.e., that if $S_1 = |C_2 - C_1|$ and $S_{n+1} = S_n + |C_{n+2} - C_{n+1}|$ for each positive integer n, then the sequence $S_1, S_2, S_3, \ldots$ converges.

By Theorem 13, $S_1, S_2, S_3, \ldots$ has a sequential limit S. $S_1, S_2, S_3, \ldots$ is non-decreasing so S must be not less than S_n for each positive integer n.

Therefore, $1 + S - S_n$ is a positive number for each positive integer n, so for each positive integer n, let K_n denote the disk with center C_n and radius $1 + S - S_n$. $|C_{n+2} - C_{n+1}| = S_{n+1} - S_n = (1 + S - S_n) - (1 + S - S_{n+1})$, so by Theorem 10, K_{n+2} is a subset of K_{n+1}. If K_1 denotes the disk with center C_1 and radius $1 + S$, $|C_2 - C_1| = S_1 = (1 + S) - (1 + S - S_1)$, so by Theorem 10, K_2 is a subset of K_1. Therefore, K_1, $K_2, K_3, \ldots$ is a nest of disks having centers $C_1, C_2, C_3, \ldots$.

Suppose there is a nest $K_1, K_2, K_3, \ldots$ of disks such that C_n is the center of K_n for each positive integer n. $S_1 = |C_2 - C_1| \leq r_1 - r_2$ $S_2 = S_1 + |C_3 - C_2| \leq (r_1 - r_2) + (r_2 - r_3) = r_1 - r_3$, $S_3 = S_2 + |C_4 - C_3| \leq (r_1 - r_3) + (r_3 - r_4) = r_1 - r_4$, ...

$S_{n+1} = S_n + |C_{n+2} - C_{n+1}| \leq (r_1 - r_{n+1}) + (r_{n+1} - r_{n+2}) = r_1 - r_{n+2}$

$r_1, r_2, r_3, \ldots$ is a non-increasing positive number sequence so by Theorem 7 there is a non-negative real number r such that r is the sequential limit of $r_1, r_2, r_3, \ldots$. For each positive integer n, $r_n \leq r$, so $r_1 - r_n \leq r_1 - r$.

$S_{n+1} = S_n + |C_{n+2} - C_{n+1}| \leq r_1 - r_{n+2} \leq r_1 - r$, so $S_n \leq S_{n+1} \leq r_1 - r$, and, by Corollary 7a, there is a real number S which is the sequential limit of $S_1, S_2, S_3, \ldots$.

<u>Theorem 15.</u> Suppose that $C_1, C_2, C_3, \ldots$ is a number sequence such that $C_{n+1} \neq C_n$, n = 1, 2, 3, ... For $C_1, C_2, C_3, \ldots$ to converge absolutely, it is necessary and sufficient that there be a positive number sequence $d_1, d_2, d_3, \ldots$ such that $\dfrac{|C_{n+2} - C_{n+1}|}{|C_{n+1} - C_n|} \leq \dfrac{d_n}{1 + d_{n+1}}$, n = 1,2,3,... .

Moreover, if $d_1, d_2, d_3, \ldots$ is such a sequence and C is the sequential limit of $C_1, C_2, C_3, \ldots$ then $|C_{n+1} - C| \leq d_n |C_{n+1} - C_n|$, n = 1, 2, 3, ...

<u>Proof:</u> Suppose $S_1 = |C_2 - C_1|$ and $S_{n+1} = S_n + |C_{n+2} - C_{n+1}|$, n = 1,2,3, ...

Suppose there is a positive number sequence $d_1, d_2, d_3, \ldots$ such that

$\frac{|C_{n+2} - C_{n+1}|}{|C_{n+1} - C_n|} \leq \frac{d_n}{1 + d_{n+1}}$. Then, if n = 1, $\frac{d_1}{1 + d_2} \geq \frac{S_2 - S_1}{S_1}$, and if n is an integer greater than 1, $\frac{d_n}{1 + d_{n+1}} \geq \frac{S_{n+1} - S_n}{S_n - S_{n-1}}$. $C_{n+1} \neq C_n$, so S_1, S_2, S_3, ... is an increasing number sequence, i.e., $S_{n+1} - S_n > 0$, for each positive integer n. $d_1 S_1 \geq (1 + d_2)(S_2 - S_1) = S_2 - S_1 + d_2(S_2 - S_1)$, $d_2(S_2 - S_1) \geq (1 + d_3)(S_3 - S_2) = S_3 - S_2 + d_3(S_3 - S_2)$, so $d_1 S_1 \geq S_3 - S_1 + d_3(S_3 - S_2)$, $d_3(S_3 - S_2) \geq (1 + d_4)(S_4 - S_3) = S_4 - S_3 + d_4(S_4 - S_3)$, so $d_1 S_1 \geq S_4 - S_1 + d_4(S_4 - S_3)$, and if $d_1 S_1 \geq S_{n-1} - S_1 + d_{n-1}(S_{n-1} - S_n)$, $d_{n-1}(S_{n-1} - S_{n-2}) \geq (1 + d_n)(S_n - S_{n-1}) = S_n - S_{n-1} + d_n(S_n - S_{n-1})$, so $d_1 S_1 \geq S_n - S_1 + d_n(S_n - S_{n-1})$, and $d_n(S_n - S_{n-1}) > 0$. Therefore, $(1 + d_1)S_1 > S_n$ for each positive integer n, and so by Corollary 7a S_1, S_2, S_3, ... has a sequential limit S. By Theorem 9, the sequence S_1, S_2, S_3, . . . converges, so C_1, C_2, C_3, ... is absolutely convergent.

Suppose that the sequence C_1, C_2, C_3, ... is absolutely convergent, i.e., that the sequence S_1, S_2, S_3, ... is convergent. By Theorem 13, S_1, S_2, S_3, ... has a sequential limit S, and S_1, S_2, S_3, ... is increasing so $S > S_n$ for each positive integer n.

If for each integer n greater than 1, $d_n = \frac{S - S_n}{S_n - S_{n-1}}$, then $d_n > 0$ and $d_{n+1} = \frac{S - S_{n+1}}{S_{n+1} - S_n}$. $1 + d_{n+1} = \frac{S_{n+1} - S_n}{S_{n+1} - S_n} + \frac{S - S_{n+1}}{S_{n+1} - S_n} = \frac{S - S_n}{S_{n+1} - S_n}$.

Therefore, $\dfrac{d_n}{1+d_{n+1}} = \dfrac{S - S_n}{S_n - S_{n-1}} \cdot \dfrac{S_{n+1} - S_n}{S - S_n} = \dfrac{S_{n+1} - S_n}{S_n - S_{n-1}} = \dfrac{|C_{n+2} - C_{n+1}|}{|C_{n+1} - C_n|}$

If $d_1 = \dfrac{S - S_1}{S_1}$ then $d_1 > 0$, and $1 + d_2 = \dfrac{S - S_1}{S_2 - S_1}$ so $\dfrac{d_1}{1 + d_2} =$

$\dfrac{S - S_1}{S_1} \cdot \dfrac{S_2 - S_1}{S - S_1} = \dfrac{S_2 - S_1}{S_1} = \dfrac{|C_3 - C_2|}{|C_2 - C_1|}$.

Therefore, there is a positive number sequence $d_1, d_2, d_3, \ldots$ such that

$\dfrac{d_n}{1 + d_{n+1}} \geq \left|\dfrac{C_{n+2} - C_{n+1}}{C_{n+1} - C_n}\right|$, $n = 1,2,3, \ldots$.

Suppose C denotes the sequential limit of $C_1, C_2, C_3, \ldots$.

If n is a positive integer, $d_n \left|C_{n+1} - C_n\right| \geq (1 + d_n) \left|C_{n+2} - C_{n+1}\right|$ $= \left|C_{n+2} - C_{n+1}\right| + d_n \left|C_{n+2} - C_{n+1}\right|$

$d_{n+1} \left|C_{n+2} - C_{n+1}\right| \geq (1 + d_{n+2}) \left|C_{n+3} - C_{n+2}\right| = \left|C_{n+3} - C_{n+2}\right| + d_{n+2} \left|C_{n+3} - C_{n+2}\right|$

so, $d_n \left|C_{n+1} - C_n\right| \geq \left|C_{n+2} - C_{n+1}\right| + \left|C_{n+3} - C_{n+2}\right| + d_{n+2} \left|C_{n+3} - C_{n+2}\right|$, but $\left|C_{n+2} - C_{n+1}\right| + \left|C_{n+3} - C_{n+2}\right| \geq \left|C_{n+3} - C_{n+1}\right|$, so $d_n \left|C_{n+1} - C_n\right| \geq \left|C_{n+3} - C_{n+1}\right| + d_{n+2} \left|C_{n+3} - C_{n+2}\right|$.

$d_{n+2} \left|C_{n+3} - C_{n+2}\right| \geq (1 + d_{n+3}) \left|C_{n+4} - C_{n+3}\right| = \left|C_{n+4} - C_{n+3}\right| + d_{n+3} \left|C_{n+4} - C_{n+3}\right|$ so $d_{n+1} \left|C_{n+1} - C_n\right| \geq \left|C_{n+3} - C_{n+1}\right| + \left|C_{n+4} - C_{n+3}\right| + d_{n+3} \left|C_{n+4} - C_{n+3}\right| \geq \left|C_{n+4} - C_{n+1}\right| + d_{n+3} \left|C_{n+4} - C_{n+3}\right|$.

By continuing this process, we find that if k is a positive integer, that $d_n \left|C_{n+1} - C_n\right| \geq \left|C_{n+1+k} - C_{n+1}\right| + d_{n+k} \left|C_{n+1+k} - C_{n+k}\right|$.

d_{n+k} is a positive number, so $d_{n+k}\left|C_{n+1+k} - C_{n+k}\right|$ is positive, so $d_n\left|C_{n+1} - C_n\right| \geq \left|C_{n+1+k} - C_{n+1}\right|$ $\left|C_{n+1+k} - C_{n+1}\right| = \left|C_{n+1} - C_{n+1+k}\right| + \left|C_{n+1+k} - C\right| - C_{n+1+k} - C\left| \geq \right|C_{n+1+k} - C\left| - \right|C_{n+1+k} - C|$,

so, $d_n\left|C_{n+1} - C_n\right| \geq \left|C_{n+1} - C\right| - \left|C_{n+1+k} - C\right|$, or, $d_n\left|C_{n+1} - C_n\right| + \left|C_{n+1+k} - C\right| \geq \left|C_{n+1} - C\right|$.

Let q denote the least non-negative number such that $d_n\left|C_{n+1} - C_n\right| + q \geq \left|C_{n+1} - C\right|$. Then if k is a positive integer, $\left|C_{n+1+k} - C\right| \geq q$. C is the sequential limit of $C_1, C_2, C_3, \ldots$ so if $p > O$, there exists a positive number N such that if m is an integer greater than N, $p > \left|C_m - C\right|$. If m is greater than N, $n+1+m > N$, so $p > \left|C_{n+1+m} - C\right| \geq q$, i.e., if $p > O$, $p > q$. Therefore, by Theorem 3, $q = O$ and $d_n\left|C_{n+1} - C_n\right| \geq \left|C_{n+1} - C\right|$.

VITA

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