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NUMBER SEQUENCES

by

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THESIS

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For the Degree of

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P R E F A C E

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R. B. S.

The University of Texas
Austin, Texas
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A B S T R A C T

Some preliminary theorems about numbers are proved, and some particular types of number sequences are shown to have sequential limits. It is then shown that if a sequence has a sequential limit, that sequence is convergent, and that if a sequence is convergent, it has a sequential limit; two necessary and sufficient conditions for absolute convergence of a number sequence are established.

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CHAPTER I

CERTAIN PROPERTIES OF NUMBERS

The following five theorems are intended to establish certain properties of real numbers which will be useful in proving the later theorems concerning sequences.

Theorem 1. If p is a positive number, then there is an integer q such that $q > p$.

Proof: Suppose p is a number such that if q is an integer, then $q < p$.

Let Q denote the set such that X belongs to Q if and only if X is an integer. Q has upper bound p, so Q has a least upper bound u.

If $e > o$, $u - e < u$ so there exists a positive integer n such that $n > u - e$; in particular $\frac{1}{2} > o$ so there exists an integer n such that $n > u - \frac{1}{2}$; so $n + \frac{1}{2} > u$.

But $n + 1$ is an integer, and $n + 1 > n + \frac{1}{2}$, so $n + 1 > u$, i.e., u is not an upper bound of Q. Therefore, the supposition that Q has an upper bound p must be false.

Theorem 2. If $p > o$, then there is a positive integer n such that $1/n < p$.

Proof: Suppose there is a number p such that if n is a positive integer, $1/n > p$. Then if n is a positive integer, $1/p > n$; but ac-

cording to Theorem 1, there is an integer m such that $m > 1/p$. Therefore, the supposition that there is such a number as p must be false.

Theorem 3. Suppose that p is a non-negative real number such that if $q > o$, then $p < q$. Then $p = o$.

Proof: Either $p = o$, or $p > o$.

If $p > o$, then p satisfies the requirement on q in the hypothesis of this theorem, so $p < p$. This cannot be true, so $p = o$.

Theorem 4. If $p > o$ and n is a positive integer, then $(1 + p)^n \geq 1 + np$.

Proof: $(1 + p)^1 = 1 + 1 \cdot p$

$(1 + p)^2 = 1 + 2p + p^2$ and $p^2 > o$,

so $(1 + p)^2 > 1 + 2p$

$(1 + p)^3 = (1 + p)(1 + p)^2 > (1 + p)(1 + 2p) = 1 + 3p + 2p^2$

so $(1 + p)^3 > 1 + 3p$

and, continuing this process, for each positive integer n if $(1 + p)^{n-1} \geq 1 + (n - 1)p$, $(1 + p) > o$, so

$$(1 + p)^n \geq (1 + p)[1 + (n - 1)p] = 1 + np + (n - 1)p^2$$

and $(n - 1)p^2 \geq o$,

so $(1 + p)^n \geq 1 + np$

Theorem 5. If $O < r < 1$ and $p > O$, then there is a positive number N such that $O < r^n < p$ if n is an integer greater than N.

Proof: $r < 1$, so $1 < 1/r$, i.e., there is a positive number k such that $1/r = 1 + k$. Therefore, if n is a positive integer, $(1/r)^n = 1/r^n = (1 + k)^n$,

and, by Theorem 4, $(1 + k)^n \geq 1 + nk$.

$1/kp > 0$, so, by Theorem 1, there is a positive integer n such that $n > 1/kp$. If $n > 1/kp$, $nk > 1/p$, so $1 + nk > 1 + 1/p$. Therefore, $1/r^n > 1 + 1/p$. $1 + 1/p > 1/p$, so $1/r^n > 1/p$, or $p > r^n$. Moreover, for each positive integer n, r^n is the product of n positive numbers, so $r^n > 0$. Therefore, if N denotes the number $1/kp$ and n is an integer greater than N, $0 < r^n < p$.

CHAPTER II

SOME PARTICULAR SEQUENCES WHICH HAVE SEQUENTIAL LIMITS

Definition 1. Suppose that $S_1, S_2, S_3, \ldots$ is a number-sequence. The statement that S is the sequential limit of $S_1, S_2, S_3, \ldots$ means that S is a number and that if $p > 0$, then there is a positive number N such that $|S_n - S| < p$ if n is an integer greater than N.

Lemma 6a. If r is a number, and $r \neq 1$, and n is a positive integer, then $1/(1-r) = 1 + r + r^2 + r^3 + \ldots + r^{n-1} + r^n/(1-r)$.

Proof: $1/(1-r) = (1-r + r)/(1-r)$

$= 1 + r/(1-r)$

$= 1 + (r - r^2 + r^2)/(1-r)$

$= 1 + r + r^2/(1-r)$, and, continuing this process, $r^{n-1}/(1-r)$

$= (r^{n-1} - r^n + r^n)/(1-r)$

$= (r^{n-1}(1-r) + r^n)/(1-r)$

$= r^{n-1} + r^n/(1-r)$ for each positive integer n, so

$$1/(1-r) = 1 + r + r^2 + r^3 + \ldots + r^{n-1} + r^n/(1-r)$$

Theorem 6. If $|r| < 1$ and $p > 0$, then there is a positive number N such that $\left|(1 + r + r^2 + \ldots + r^{n-1}) - 1/(1-r)\right| < p$ if n is an integer greater than N.

Proof: By Lemma 6a, $1 + r + r^2 + \ldots + r^{n-1} + r^n/(1-r) = 1/(1-r)$

so, $(1 + r + r^2 + \ldots + r^{n-1}) - 1/(1-r) = r^n/(1-r)$.

Therefore, $\left|(1 + r + r^2 + \ldots + r^{n-1}) - 1/(1-r)\right| = \left|r^n/(1-r)\right|$.

$0 < |r| < 1$ and $|1-r|\,p > 0$, so, by Theorem 5, there exists a positive number N such that $0 < |r|^n < |1-r|\,p$ if n is an integer greater than N.

$\left|\frac{r^n}{1-r}\right| = \frac{|r^n|}{|1-r|} = \frac{|r|^n}{|1-r|}$, so if n is an integer greater than N, $|r|^n < |1-r|\,p$ and $\left|\frac{r^n}{1-r}\right| = \frac{|r|^n}{|1-r|} < \frac{|1-r|\,p}{|1-r|} = p$. Therefore, $\left|(1 + r + r^2 + \ldots + r^{n-1}) - 1/(1-r)\right| < p$, or $1/(1-r)$ is the sequential limit of the sequence 1, $1 + r$, $1 + r + r^2$, $1 + r + r^2 + r^3$, ...

<u>Theorem 7.</u> Suppose that $S_1, S_2, S_3, \ldots$ is a non-increasing positive-number sequence (i.e., $0 < S_{n+1} \leq S_n$ for each positive integer n). There is a non-negative real number S which is the sequential limit of $S_1, S_2, S_3, \ldots$.

<u>Proof:</u> $S_1, S_2, S_3, \ldots$ has lower bound zero, so there is a greatest number S which exceeds no element of $S_1, S_2, S_3, \ldots$. If $p > 0$, there exists a positive integer N such that $S_N < S + p$; otherwise, $S + P$ would be a lower bound of $S_1, S_2, S_3, \ldots$ which exceeds S, the greatest lower bound of $S_1, S_2, S_3, \ldots$.

If $n > N$, $S_N \geq S_n > S$ and $S \geq 0$, so $S_N - S \geq S_n - S > 0$. But $p > S_N - S$ so $p > S_n - S > 0$, and $|S_n - S| < p$ for each integer n greater than N. Therefore, S is a sequential limit of $S_1, S_2, S_3, \ldots$.

<u>Corollary 7a.</u> Suppose that $S_1, S_2, S_3, \ldots$ is a bounded non-decreasing number sequence (i.e., that there is a positive number B such that $S_n \leq S_{n+1} \leq B$ for each positive integer n). There is a real number S which is the sequential limit of $S_1, S_2, S_3, \ldots$.

Proof: Let B' denote $B+1$. Then, $B'>B$, so if $S_n \leq S_{n+1} \leq B$, $S_n \leq S_{n+1} < B'$.

For each positive integer n, $S_n \leq S_{n+1} < B'$, so $-S_n \geq -S_{n+1} > -B'$ and $B'-S_n \geq B'-S_{n+1} > 0$, so $B' - S_1$, $B' - S_2$, $B' - S_3$, ... is a non-increasing positive number sequence. Therefore, by Theorem 7, there is a non-negative real number k which is the sequential limit of $B' - S_1$, $B' - S_2$, $B' - S_3$, ...

If $p > 0$, there is a positive number N such that if n is an integer greater than N, $|B' - S_n - k| < p$. Let S denote $B - k$. Then $-k = S - B$ so $|B' - S_n - k| = |B' - S_n + S - B'| = |S_n - S|$. Therefore, if $n > N$, $|S_n - S| < p$; that is, S is the sequential limit of S_1, S_2, S_3, ...

<u>Theorem 8</u>. Suppose that S_1, S_3, S_5,... is a non-increasing number sequence and that S_2, S_4, S_6,... is a non-decreasing number sequence. If S_1, S_2, S_3,... has a sequential limit S, then $S_{2n-1} \geq S \geq S_{2n}$ for each positive integer n. If $S_{2n-1} \geq S_{2n}$ for each positive integer n, then for S_1, S_2, S_3,... to have a sequential limit it is necessary and sufficient that the following statement be true: if $P > 0$, then there is a positive integer k such that $|S_k - S_{k+1}| < p$.

<u>Proof:</u> Suppose T is a number and, for some positive integer n, $T > S_{2n-1}$. S_1, S_3, ... , S_{2n-1}, ... is a non-increasing number sequence so if k is an integer greater than n, $S_{2n-1} \geq S_{2k-1}$. If $p = T - S_{2n-1}$, $p > 0$, and if N is a positive number there is an integer k such that $k > 1/2\ (N + 2n)$, (i.e., $2k - 1 > N + 2n - 1$). If k is such an integer, $2k - 1 > N$, but $|S_{2k-1} - T| > T - S_{2n-1}$. Therefore, for each positive number N there is an integer 2k-1 greater than N such that $|S_{2k-1} - T| > p$. So if S is a sequential limit of S_1, S_2, S_3,... it must be true that $S_{2n+1} \geq S$ for each positive integer N. A similar argument shows that if S is a sequential limit of S_1, S_2, S_3,..., $S \geq S_{2n}$ for each positive integer n.

Suppose that $S_1, S_2, S_3, \ldots$ has a sequential limit S, and $p > 0$. $p/2 > 0$, so there is a positive number N such that if k is an integer greater than N, $|S_k - S| < p/2$. If $k > N$, $k+1 > N$ so $|S_{k+1} - S| < p/2$. $|S_k - S_{k+1}| = |(S_k - S) - (S_{k+1} - S)| \leq |S_k - S| + |S_{k+1} - S| < p/2 + p/2$, so $|S_k - S_{k+1}| < p$ if $k > N$.

Suppose if $p > 0$ there is a positive integer k such that $|S_k - S_{k+1}| < p$ and $S_{2n+1} \geq S_{2n}$ for each positive integer n.

$S_2, S_4, S_6, \ldots$ is a non-decreasing real number sequence, and for each n, $S_{2n+1} \geq S_{2n}$.

$S_1 \geq S_{2n+1}$ since $S_1, S_3, S_5, \ldots$ is non-increasing. Therefore, $S_{2n} \leq S_{2n+2} \leq S_1$ and so, by Corollary 7a, $S_2, S_4, S_6, \ldots$ has a sequential limit S.

There is a positive number N' such that $|S_{2n} - S| < p/2$, if n is an integer greater than $N'/2$. By hypothesis there is a positive integer k' such that $|S_{k'} - S_{k'+1}| < p/2$.

Suppose m is an integer greater than $k' + N'$. If k' is odd and m is odd, $S_{k'} \geq S_m$ since $S_1, S_3, S_5, \ldots$ is non-increasing and $S_{k'+1} \leq S_{m+1}$ since $S, S_2, S_4, S_6, \ldots$ is non-decreasing. From the hypothesis of the theorem we know that $S_m \geq S_{m+1}$, so $|S_m - S_{m+1}| = S_m - S_{m+1} \leq S_{k'} - S_{k'+1} = |S_{k'} - S_{k'+1}| < p/2$. $|S_m - S| \leq |S_m - S_{m+1}| + |S_{m+1} - S|$ and $m+1$ is an even integer greater than N' so $|S_{m+1} - S| < p/2$ and $|S_m - S| < p/2 + p/2$, or $|S_m - S| < p$.

If k' is even and m is odd, $S_{k'} \geq S_{m+1}$ because $S_1, S_3, S_5, \ldots$ is non-increasing, and $S_{k'+1} \leq S_m$ because $S_2, S_4, S_6, \ldots$ is non-decreasing. $S_m \geq S_{m+1}$, so $|S_m - S_{m+1}| = S_m - S_{m+1} \leq S_{k'+1} - S_{k'} = |S_{k'} - S_{k'+1}| < p/2$. $|S_m - S| \leq |S_m - S_{m+1}| + |S_{m+1} - S|$ and $m+1$ is an even integer greater than n so $|S_{m+1} - S| < p/2$, and $|S_m - S| < p/2 + p/2$, so $|S_m - S| < p$.

Therefore, if $p > 0$ and m is an integer greater than $N' + k'$, $|S_m - S| < p$, so S is a sequential limit of $S_1, S_2, S_3, \ldots$.

CHAPTER III

CONVERGENCE AND SEQUENTIAL LIMITS

Definition 2. Suppose that $S_1, S_2, S_3, \ldots$ is a number sequence. The statement that $S_1, S_2, S_3, \ldots$ converges means that if p is a positive number then there is a positive number N such that $|S_m - S_n| < p$ if m and n are integers greater than N.

Theorem 9. If the sequence $S_1, S_2, S_3, \ldots$ has a sequential limit, then the sequence converges.

Proof: By hypothesis, $S_1, S_2, S_3, \ldots$ has a sequential limit S, i.e., if $p > 0$ there exists a positive number N such that $|S_n - S| < p$ if n is an integer greater than N.

If $p > 0$, $p/2 > 0$, so there exists a positive number N' such that if n is an integer greater than N', $|S_n - S| < p/2$. Suppose m and n are integers greater than N'. Then, $|S_m - S_n| \leq |S_m - S| + |S_n - S|$. $|S_m - S| < p/2$ and $|S_n - S| < p/2$, so $|S_m - S_n| < p$, if m and n are integers greater than N'.

Definition 3. The statement that K is a disk with center c and radius r means that c is a number, r is a positive number, and K is a number set such that the number Z is in K if and only if $|Z - c| \leq r$.

Theorem 10. Suppose that K_1 is the disc with center c_1 and radius r_1. For K_2 to be a subset of K_1 (that is to say, for each member of K_2 to be a member of K_1,) it is necessary and sufficient that $|C_1 - C_2| \leq r_1 - r_2$.

<u>Proof</u>: Suppose $|C_1 - C_2| \leq r_1 - r_2$ and Z is in K_2, i.e., $|Z - C_2| \leq r_2$, Then $|Z - C_1| = |Z - C_2 + C_2 - C_1|$, but $|Z - C_2 + C_2 - C_1| \leq |Z - C_2| + |C_2 - C_1|$, and $|Z - C_2| \leq r_2$ and $|C_2 - C_1| = |C_1 - C_2| \leq r_1 - r_2$, so $|Z - C_1| \leq r_2 + r_1 - r_2$, $|Z - C_1| \leq r_1$, i. e., K_2 is a subset of K.

Suppose K_2 is a subset of K_1, i.e., if $|Z - C_2| \leq r_2$, then $|Z - C_1| \leq r_1$. Then $\left|\left(C_2 + r_2 \frac{C_2 - C_1}{|C_2 - C_1|}\right) - C_2\right| = r_2$, so $\left|\left(C_2 + r_2 \frac{C_2 - C_1}{|C_2 - C_1|}\right) - C_1\right| \leq r_1$, but $\left|\left(C_2 + r_2 \frac{C_2 - C_1}{|C_2 - C_1|}\right) - C_1\right| = \left|(C_2 - C_1) + (C_2 - C_1)\frac{r_2}{|C_2 - C_1|}\right|$ $= |C_2 - C_1| \cdot \left|1 + \frac{r_2}{|C_2 - C_1|}\right| = \frac{|C_2 - C_1|}{|C_2 - C_1|} \cdot \left||C_2 - C_1| + r_2\right| = \left||C_2 - C_1| + r_2\right| \leq r_1$. Therefore, since $\left(|C_2 - C_1| + r_2\right) > 0$, $\left||C_2 - C_1| + r_2\right| = |C_2 - C_1| + r_2$, so $|C_2 - C_1| + r_2 \leq r_1$, or, $|C_2 - C_1| \leq r_1 - r_2$.

<u>Definition 4</u>. The statement that K_1, K_2, K_3, ... is a nest of discs means that if n is a positive integer, then K_n is a disc and K_{n+1} is a subset of K_n.

<u>Theorem 11</u>. Suppose that $K_1, K_2, K_3, \ldots$ is a nest of disks. There is a number c such that if n is a positive integer, c is in K_n.

<u>Proof</u>: For each positive n, let r_n denote the radius and c_n denote the center of K_n. By Theorem 10, $|c_r - c_{n+1}| \leq r_n - r_{n+1}$ for each positive integer n, so $r_1, r_2, r_3, \ldots$ is a non-increasing positive number sequence. Therefore, by Theorem 7, $r_1, r_2, r_3, \ldots$ has a non-negative sequential limit r.

Suppose $r > 0$. Then $r/2 > 0$, so there exists a positive number N_1 such that if n is an integer greater than N_1, $|r_n - r| < r/2$. Further, by Theorem 9, $r_1, r_2, r_3, \ldots$ converges, so there exists a positive number N_2 such that if m and n are integers greater than N_2, $|r_m - r_n| < r/2$. Let N denote $N_1 + N_2$. Then if $m > n > N$, K_m is a subset of K_n, so, by Theorem 10, $|c_n - c_m| \leq r_n - r_m < r/2$. But $|r_m - r| < r/2$, or $r - r/2 \leq r_m \leq r + r/2$, so $r/2 \leq r_m$. Therefore, if n′ denotes a particular integer greater than N, then $|c_{n'} - c_m| \leq r_m$ for each positive integer m greater than n′; that is, $c_{n'}$ belongs to K_m for each positive integer m greater than n′. But $c_{n'}$ belongs to $K_{n'}$ and $K_{n'}$ is a subset of each element of the sequence $K_1, K_2, K_3, \ldots$ which precedes it, so for each positive integer n, $c_{n'}$ belongs to K_n.

Suppose $r = 0$. If Z is a number, there is a real number x and there is a real number y such that $Z = x + iy$. It is also true that $|Z| \geq |x|$ and $|Z| \geq |y|$. For each positive integer n, let a_n denote a real number and b_n denote a real number such that $c_n = a_n + b_n i$.

For each positive integer n, K_{n+1} is a subset of K_n, so $|c_n - c_{n+1}| \leq r_n - r_{n+1}$. $|c_n - c_{n+1}| = |(a_n - a_{n+1}) + (b_n - b_{n+1})i|$, so $|a_n - a_{n+1}| \leq |c_n - c_{n+1}| \leq r_n - r_{n+1}$, or $a_n - (r_n - r_{n+1}) \leq a_{n+1} \leq a_n + (r_n - r_{n+1})$. Therefore, $a_n - r_n \leq a_{n+1} - r_{n+1}$, and $a_{n+1} + r_{n+1} \leq a_n + r_n$.

Let S_{2n-1} denote $a_n + r_n$ and S_{2n} denote $a_n - r_n$. Then if $p > 0$, $p/2 > 0$, and the number sequence $r_1, r_2, r_3, \ldots$ has sequential limit 0, so there exists a positive number N, such that if k is an integer greater than N, $|r_k - 0| = |r_k| < p/2$. Suppose k is such an integer. Then $|S_{2k} - S_{2k-1}| = |(a_k - r_k) - (a_k + r_k)| = |-2r_k| < p$. For each positive integer n, $r_n > 0$, so $S_{2k-1} = a_k + r_k \geq a_k - r_k = S_{2k}$. Therefore, the hypothesis of Theorem 8 is satisfied, and $S_1, S_2, S_3, \ldots$ has a sequential limit a.

There exists a positive number N_2 such that if n is an integer greater than N_2, $|S_n - a| < p/2$. Suppose $N = 2(N_1 + N_2)$ and m' is an integer greater than N. Then there is an integer m greater than $N_1 + N_2$ such that either $m' = 2m$ or $m' = 2m-1$.

If $m' = 2m$, $|S_{m'} - a| = |a_m - r_m - a| \geq |a_m - a| - |r_m|$, so $|a_m - a| \leq |S_{m'} - a| + |r_m| < p/2 + p/2 = p$. If $m' = 2m-1$, $|S_{m'} - a| = |a_m + r_m - a| \geq |a_m - a| - |r_m|$, so $|a_m - a| \leq |S_{m'} - a| + |r_m| < p$. Therefore, a is a sequential limit of $a_1, a_2, a_3, \ldots$. This same argument can be used to show that $b_1, b_2, b_3, \ldots$ has a sequential limit b.

Suppose $c = a + bi$. Then if $p > 0$, there is a positive number N_1 such that if n is an integer greater than N_1, $|a_n - a| < p/2$, and there is a positive number N_2 such that if n is an integer greater than N_2, $|b_n - b| < p/2$, so if n is an integer greater than $N_1 + N_2$, $|c_n - c| = |(a_n - a) + (b_n - b)i| \leq |a_n - a| + |b_n - b| < p/2 + p/2 = p$, so c is a sequential limit of $c_1, c_2, {}^3{}_3, \ldots$.

Suppose c_m is the center of some disk K_n of the nest of disks K_1, K_2, K_3, ... and that c does not belong to K_n. Then $|c_n - c| = r_n + q$, where $q > 0$. $|c_n - c_m| + |c_m - c| \geq |c_n - c|$, so $|c_n - c_m| + |c_m - c| \geq r_n + q$.

If m is an integer greater than n, K_m is a subset of K_n so $|c_n - c_m| \leq r_n$, and $r_m + |c_m - c| \geq r_n + q$, so $|c_m - c| \geq q$. Therefore, if c did not belong to some disk of the nest K_1, K_2, K_3, ... c would not be the sequential limit of c_1, c_2, c_3, ...

<u>Theorem 12</u>. If S_1, S_2, S_3, ... is a convergent number sequence, there exists a nest K_1, K_2, K_3, ... of disks such that:

1. If n is a positive integer, S_n is in K_n, and

2. If $p > 0$, there exists a positive number N such that if n is an integer greater than N and r_n is the radius of K_n, $r_n < p$.

<u>Proof</u>: For each positive integer k, $1/k > 0$, so there exists a positive integer N such that if m and n are integers greater than N, $|S_m - S_n| < 1/k$. For each positive integer k, let N_k denote one such number. Then let n_1

denote an integer greater than N_1 and for each positive integer k greater than one, let n_k denote an integer greater than $N_k + n_{k-1}$.

If n is an integer and $n_k < n \leq n_{k+1}$, let K_n denote the disk with center S_{n_k} and radius $r_n = 1/k\left[1 + 1/n\right]$.

Then, if for some integer k, n and n + 1 are both greater than n_k and less than or equal to n_{k+1}, $1/k\left[(1 + 1/n)\right] - 1/k\left[1 + 1/n+1)\right] = 1/k\left[\frac{n+1}{n} - \frac{n+2}{n+1}\right] = \frac{1}{k}\left[\frac{n^2 + 2n + 1 - n^2 - 2n}{n(n+1)}\right] = 1/k\left[1/n(n+1)\right] > \left|S_{n_k} - S_{n_k}\right|$, so K_{n+1} is a subset of K_n.

If for some integer k, $n = n_k$, then $1/(k-1)\left[(1 + 1/n)\right] - 1/k\left[(1 + 1/n+1)\right] = 1/k-1\left[n+1/n\right] - 1/k\left[(n+2)/(n+1)\right]$, then $\left|S_{n_k} - S_{n+1}\right| < 1/(k-1)$ and $r_n - r_{n+1} = 1/(k-1)\left[1 + 1/n\right] - 1/k\left[1 + 1/(n+1)\right] = 1/(k-1)\left\{(n+1)/n\right] - 1/k\left[(n+2)/(n+1)\right]$

$$= \frac{k(n+1)^2 - n(n+2)(k-1)}{(k-1)nk} = \frac{1}{k-1}\,\frac{n^2 + 2n + k}{nk} = \frac{1}{k-1}\left(\frac{n}{k} + \frac{2}{k} + \frac{1}{nk}\right).$$

$n \geq k$, so $r_n - r_{n+1} \geq 1/(k-1) > \left|S_n - S_{n+1}\right|$, so K_{n+1} is a subset of K_n. Therefore, $K_1, K_2, K_3, \ldots$ is a nest of disks.

<u>Theorem 13.</u> Suppose that the number sequence $S_1, S_2, S_3, \ldots$ converges. There is a number S which is the sequential limit of $S_1, S_2, S_3, \ldots$.

<u>Proof:</u> By Theorem 12, there is a nest of disks $K_1, K_2, K_3, \ldots$ such that if n is a positive integer, S_n is in K_n, and such that if $p > 0$, there is a positive number N such that if n is an integer greater than N and r_n is the radius of K_n, then $r_n < p$.

By Theorem 11, there is a number S such that if n is a positive integer, S is in K_n.

Therefore, if $p > 0$, and N denotes a positive number such that if n is an integer greater than N, $r_n < p$, then $|S_n - S| < p$. Therefore, S is a sequential limit of $S_1, S_2, S_3, \ldots$.

CHAPTER IV

ABSOLUTE CONVERGENCE

Definition 5. Suppose that $C_1, C_2, C_3, \ldots$ is a number sequence. The statement that $C_1, C_2, C_3, \ldots$ converges absolutely means that if $S_1 = |C_2 - C_1|$ and $S_{n+1} = S_n + |C_{n+1} - C_{n+1}|$ for each positive integer n, then the sequence $S_1, S_2, S_3, \ldots$ converges.

Theorem 14. For the number sequence $C_1, C_2, C_3, \ldots$ to converge absolutely, it is necessary and sufficient that there be a nest, $K_1, K_2, K_3, \ldots$ of disks such that C_n is the center of K_n, $n = 1,2,3,\ldots$.

Proof: Suppose the sequence $C_1, C_2, C_3, \ldots$ converges absolutely, i.e., that if $S_1 = |C_2 - C_1|$ and $S_{n+1} = S_n + |C_{n+2} - C_{n+1}|$ for each positive integer n, then the sequence $S_1, S_2, S_3, \ldots$ converges.

By Theorem 13, $S_1, S_2, S_3, \ldots$ has a sequential limit S. $S_1, S_2, S_3, \ldots$ is non-decreasing so S must be not less than S_n for each positive integer n.

Therefore, $1 + S - S_n$ is a positive number for each positive integer n, so for each positive integer n, let K_n denote the disk with center C_n and radius $1 + S - S_n$. $|C_{n+2} - C_{n+1}| = S_{n+1} - S_n = (1 + S - S_n) - (1 + S - S_{n+1})$, so by Theorem 10, K_{n+2} is a subset of K_{n+1}. If K_1 denotes the disk with center C_1 and radius $1 + S$, $|C_2 - C_1| = S_1 = (1 + S) - (1 + S - S_1)$, so by Theorem 10, K_2 is a subset of K_1. Therefore, $K_1, K_2, K_3, \ldots$ is a nest of disks having centers $C_1, C_2, C_3, \ldots$.

Suppose there is a nest $K_1, K_2, K_3, \ldots$ of disks such that C_n is the center of K_n for each positive integer n. $S_1 = |C_2 - C_1| \leq r_1 - r_2$ $S_2 = S_1 + |C_3 - C_2| \leq (r_1 - r_2) + (r_2 - r_3) = r_1 - r_3$, $S_3 = S_2 + |C_4 - C_3| \leq (r_1 - r_3) + (r_3 - r_4) = r_1 - r_4$, . . .

$S_{n+1} = S_n + |C_{n+2} - C_{n+1}| \leq (r_1 - r_{n+1}) + (r_{n+1} - r_{n+2}) = r_1 - r_{n+2}$

$r_1, r_2, r_3, \ldots$ is a non-increasing positive number sequence so by Theorem 7 there is a non-negative real number r such that r is the sequential limit of $r_1, r_2, r_3, \ldots$. For each positive integer n, $r_n \leq r$, so $r_1 - r_n \leq r_1 - r$.

$S_{n+1} = S_n + |C_{n+2} - C_{n+1}| \leq r_1 - r_{n+2} \leq r_1 - r$, so $S_n \leq S_{n+1} \leq r_1 - r$, and, by Corollary 7a, there is a real number S which is the sequential limit of $S_1, S_2, S_3, \ldots$.

<u>Theorem 15.</u> Suppose that $C_1, C_2, C_3, \ldots$ is a number sequence such that $C_{n+1} \neq C_n$, n = 1, 2, 3, ... For $C_1, C_2, C_3, \ldots$ to converge absolutely, it is necessary and sufficient that there be a positive number sequence $d_1, d_2, d_3, \ldots$ such that $\dfrac{|C_{n+2} - C_{n+1}|}{|C_{n+1} - C_n|} \leq \dfrac{d_n}{1 + d_{n+1}}$, n = 1,2,3,... .

Moreover, if $d_1, d_2, d_3, \ldots$ is such a sequence and C is the sequential limit of $C_1, C_2, C_3, \ldots$ then $|C_{n+1} - C| \leq d_n |C_{n+1} - C_n|$, n = 1, 2, 3, ...

<u>Proof:</u> Suppose $S_1 = |C_2 - C_1|$ and $S_{n+1} = S_n + |C_{n+2} - C_{n+1}|$, n = 1,2,3, ...

Suppose there is a positive number sequence $d_1, d_2, d_3, \ldots$ such that

$\left|\frac{C_{n+2} - C_{n+1}}{C_{n+1} - C_n}\right| \leq \frac{d_n}{1 + d_{n+1}}$. Then, if $n = 1$, $\frac{d_1}{1 + d_2} \geq \frac{S_2 - S_1}{S_1}$, and if n is an integer greater than 1, $\frac{d_n}{1 + d_{n+1}} \geq \frac{S_{n+1} - S_n}{S_n - S_{n-1}}$; $C_{n+1} \neq C_n$, so S_1, S_2, S_3, ... is an increasing number sequence, i.e., $S_{n+1} - S_n > 0$, for each positive integer n. $d_1S_1 \geq (1 + d_2)(S_2 - S_1) = S_2 - S_1 + d_2(S_2 - S_1)$, $d_2(S_2 - S_1) \geq (1 + d_3)(S_3 - S_2) = S_3 - S_2 + d_3(S_3 - S_2)$, so $d_1S_1 \geq S_3 - S_1 + d_3(S_3 - S_2)$, $d_3(S_3 - S_2) \geq (1 + d_4)(S_4 - S_3) = S_4 - S_3 + d_4(S_4 - S_3)$, so $d_1S_1 \geq S_4 - S_1 + d_4(S_4 - S_3)$, and if $d_1S_1 \geq S_{n-1} - S_1 + d_{n-1}(S_{n-1} - S_n)$, $d_{n-1}(S_{n-1} - S_{n-2}) \geq (1 + d_n)(S_n - S_{n-1}) = S_n - S_{n-1} + d_n(S_n - S_{n-1})$, so $d_1S_1 \geq S_n - S_1 + d_n(S_n - S_{n-1})$, and $d_n(S_n - S_{n-1}) > 0$. Therefore, $(1 + d_1)S_1 > S_n$ for each positive integer n, and so by Corollary 7a S_1, S_2, S_3, ... has a sequential limit S. By Theorem 9, the sequence S_1, S_2, S_3, . . . converges, so C_1, C_2, C_3, ... is absolutely convergent.

Suppose that the sequence C_1, C_2, C_3, ... is absolutely convergent, i.e., that the sequence S_1, S_2, S_3, ... is convergent. By Theorem 13, S_1, S_2, S_3, ... has a sequential limit S, and S_1, S_2, S_3, ... is increasing so $S > S_n$ for each positive integer n.

If for each integer n greater than 1, $d_n = \frac{S - S_n}{S_n - S_{n-1}}$, then $d_n > 0$ and

$$d_{n+1} = \frac{S - S_{n+1}}{S_{n+1} - S_n} \quad . \quad 1 + d_{n+1} = \frac{S_{n+1} - S_n}{S_{n+1} - S_n} + \frac{S - S_{n+1}}{S_{n+1} - S_n} = \frac{S - S_n}{S_{n+1} - S_n}.$$

Therefore, $\dfrac{d_n}{1 + d_{n+1}} = \dfrac{S - S_n}{S_n - S_{n-1}} \cdot \dfrac{S_{n+1} - S_n}{S - S_n} = \dfrac{S_{n+1} - S_n}{S_n - S_{n-1}} = \dfrac{|C_{n+2} - C_{n+1}|}{|C_{n+1} - C_n|}$

If $d_1 = \dfrac{S - S_1}{S_1}$ then $d_1 > 0$, and $1 + d_2 = \dfrac{S - S_1}{S_2 - S_1}$ so $\dfrac{d_1}{1 + d_2} =$

$\dfrac{S - S_1}{S_1} \cdot \dfrac{S_2 - S_1}{S - S_1} = \dfrac{S_2 - S_1}{S_1} = \dfrac{|C_3 - C_2|}{|C_2 - C_1|}$.

Therefore, there is a positive number sequence $d_1, d_2, d_3, \ldots$ such that

$\dfrac{d_n}{1 + d_{n+1}} \geq \left|\dfrac{C_{n+2} - C_{n+1}}{C_{n+1} - C_n}\right|$, $n = 1,2,3, \ldots$.

Suppose C denotes the sequential limit of $C_1, C_2, C_3, \ldots$.

If n is a positive integer, $d_n |C_{n+1} - C_n| \geq (1 + d_n) |C_{n+2} - C_{n+1}|$

$= |C_{n+2} - C_{n+1}| + d_n |C_{n+2} - C_{n+1}|$

$d_{n+1} |C_{n+2} - C_{n+1}| \geq (1 + d_{n+2}) |C_{n+3} - C_{n+2}| = |C_{n+3} - C_{n+2}| + d_{n+2} |C_{n+3} - C_{n+2}|$

so, $d_n |C_{n+1} - C_n| \geq |C_{n+2} - C_{n+1}| + |C_{n+3} - C_{n+2}| + d_{n+2} |C_{n+3} - C_{n+2}|$,

but $|C_{n+2} - C_{n+1}| + |C_{n+3} - C_{n+2}| \geq |C_{n+3} - C_{n+1}|$, so $d_n |C_{n+1} - C_n| \geq |C_{n+3} - C_{n+1}| + d_{n+2} |C_{n+3} - C_{n+2}|$.

$d_{n+2} |C_{n+3} - C_{n+2}| \geq (1 + d_{n+3}) |C_{n+4} - C_{n+3}| = |C_{n+4} - C_{n+3}| + d_{n+3} |C_{n+4} - C_{n+3}|$, so $d_{n+1} |C_{n+1} - C_n| \geq |C_{n+3} - C_{n+1}| + |C_{n+4} - C_{n+3}| + d_{n+3} |C_{n+4} - C_{n+3}| \geq |C_{n+4} - C_{n+1}| + d_{n+3} |C_{n+4} - C_{n+3}|$.

By continuing this process, we find that if k is a positive integer, that $d_n |C_{n+1} - C_n| \geq |C_{n+1+k} - C_{n+1}| + d_{n+k} |C_{n+1+k} - C_{n+k}|$.

d_{n+k} is a positive number, so $d_{n+k}\left|C_{n+1+k} - C_{n+k}\right|$ is positive, so $d_n \left|C_{n+1} - C_n\right| \geq \left|C_{n+1+k} - C_{n+1}\right|$ $\left|C_{n+1+k} - C_{n+1}\right| = \left|C_{n+1} - C_{n+1+k}\right| + \left|C_{n+1+k} - C\right| - \left|C_{n+1+k} - C\right| \geq \left|C_{n+1+k} - C\right| - \left|C_{n+1+k} - C\right|$,

so, $d_n\left|C_{n+1} - C_n\right| \geq \left|C_{n+1} - C\right| - \left|C_{n+1+k} - C\right|$, or, $d_n\left|C_{n+1} - C_n\right| + \left|C_{n+1+k} - C\right| \geq \left|C_{n+1} - C\right|$.

Let q denote the least non-negative number such that $d_n\left|C_{n+1} - C_n\right| + q \geq \left|C_{n+1} - C\right|$. Then if k is a positive integer, $\left|C_{n+1+k} - C\right| \geq q$. C is the sequential limit of C_1, C_2, C_3, ... so if $p > 0$, there exists a positive number N such that if m is an integer greater than N, $p > \left|C_m - C\right|$. If m is greater than N, $n+1+m > N$, so $p > \left|C_{n+1+m} - C\right| \geq q$, i.e., if $p > 0$, $p > q$. Therefore, by Theorem 3, $q = 0$ and $d_n\left|C_{n+1} - C_n\right| \geq \left|C_{n+1} - C\right|$.

VITA

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